Lecture IX: More on Principle of Inclusion-Exclusion; Derangements

Recall: X universal set $E=\{e_1,\ldots,e_m\}$ properties, $|X|=n$

---- : $N_{\geq T} := \#\{x\in X \mid X \text{ satisfies properties in } T\}$

$N_{=T} := \#\{x\in X \mid X \text{ ---- exactly the properties in } T\}$

Def E homogeneous if $N_{\geq T} = N_{\geq T'}$ whenever $|T|=|T'|$

PIE: (1) $N_{\geq B} = \sum\limits_{T\geq B} N_{=T}$ & (2) $N_{=B} = \sum\limits_{E\geq T\geq B} (-1)^{|T|-|B|} N_{\geq T}$.

TODAY: More general version, interpreting PIE as a way of inverting a linear map

§1 General PIE:

Fix S with $|S|=n$ & $V := \{f: 2^S \longrightarrow K\}$ for K fixed field

Fact: V is a 2^n-dimensional K-vector space

General PIE: Let $\phi: V\longrightarrow V$ be the linear transformation defined by

$\phi(f): 2^S \longrightarrow K$ with $\phi(f)_{(B)} = \sum\limits_{T\geq B} f(T)$ for all $B\subseteq S$, $f\in V$

Then, ϕ is invertible & $\phi^{-1}: V\longrightarrow V$ is defined for any $g\in V$ via

$\phi^{-1}(g): 2^S \longrightarrow K$ with $\phi^{-1}(g)_{(B)} = \sum\limits_{T\geq B} (-1)^{|T-B|} g(T)$ for all $B\subseteq S$

Proof: We set $\Psi: V\longrightarrow V$ linear map with $\Psi(g)_{(B)} = \sum\limits_{T\geq B} (-1)^{|T-B|} g(T)$.

Since $\dim_K V<\infty$, to show $\Psi=\phi^{-1}$ it's enough to check $\phi\circ\Psi = id_V$.

Pick $g: 2^S\longrightarrow K$ Want to show: $\phi\circ\Psi(g) = g$ ie $\phi\circ\Psi(g)_{(B)} = g(B)$ $\forall B\subseteq S$

$$\phi\circ\underbrace{\Psi(g)}_{f\in V}(B) = \sum_{T\geq B} f(T) \underset{\text{def of } f}{=} \sum_{T\geq B}\sum_{R\geq T} (-1)^{|R-T|} g(R) \underset{\text{reorder sum}}{=} \sum_{R\geq B}\Big(\sum_{R\geq T\geq B} (-1)^{|R-T|}\Big) g(R)$$

$$= \sum_{R\geq B} (-1)^{|R|-|B|} \underbrace{\Big(\sum_{R\geq T\geq B} (-1)^{|T-B|}\Big)}_{\text{Want to compute this!}} g(R)$$ because $|R| = |R-T| + |T-B| + |B|$

Fix $m := |R|-|B|$

$$\sum_{R\geq T\geq B} (-1)^{|T-B|} \underset{\text{group all } T \text{ with same } |T|-|B| = |T-B| \rightsquigarrow \binom{m}{k} \text{ many}}{=} \sum_{k=0}^{m} (-1)^k \binom{m}{k} = (1+x)^m\Big|_{x=-1} = \begin{cases} 0 & \text{if } m>0 \\ 1 & \text{if } m=0\end{cases}$$

Then $\phi \circ \Psi(g)(B) = \sum_{R \supseteq B} (-1)^{|R|-|B|} \delta_{\emptyset, R \setminus B}\, g(R) = g(B)$. $\square$

. Dual version (interchange $\subseteq$ & $\supseteq$): Fix $\tilde{\phi}: V \longrightarrow V$ linear map defined for any $f \in V$ as $\tilde{\phi}(f): 2^S \longrightarrow K$ with $\tilde{\phi}(f)(B) = \sum_{T \subseteq B} f(T)$ for all $B \subseteq S$

Then: $\tilde{\phi}$ is invertible & $\tilde{\phi}^{-1}: V \longrightarrow V$ is defined for any $g \in V$ via $\tilde{\phi}^{-1}(g): 2^S \longrightarrow K$ by $\tilde{\phi}^{-1}(g)(B) = \sum_{T \subseteq B} (-1)^{|B \setminus T|} g(T)$ for all $B \subseteq S$.

. Original PIE dualized: (interchange $\subseteq$ & $\supseteq$) Set $N_{\subseteq T} := \{x \in X \mid x$ satisfies at most the properties in $T\}$

Then: (1) $N_{\subseteq B} = \sum_{T \subseteq B} N_{=T}$ & (2) $N_{=B} = \sum_{T \subseteq B} (-1)^{|B-T|} N_{\subseteq T}$.

§2 Homogeneous case: Assume E is homogeneous, $|X| = n$. Write:

$$N_{=B} = p(|B|) \quad \& \quad N_{\supseteq B} = q(|B|) \quad \text{for 2 functions } p, q: [n] \to \mathbb{Z}_{\geq 0}$$

Homogeneous PIE says:

$$(1)\quad q(|B|) = \sum_{T \supseteq B} p(|T|) = \sum_{k \geq |B|}^{n} p(k) \binom{n-|B|}{k-|B|} = \sum_{k=0}^{n-|B|} p(k+|B|) \binom{n-|B|}{k}$$

$$(2)\quad p(|B|) = \sum_{T \supseteq B} (-1)^{|T \setminus B|} q(|T|) = \sum_{k \geq |B|}^{n} (-1)^{k-|B|} \binom{n-|B|}{k-|B|} q(k).$$

$$= \sum_{k=0}^{n-|B|} (-1)^k \binom{n-|B|}{k} q(k+|B|)$$

Obs: New interpretation of PIE $\Rightarrow \left[\begin{matrix} j = n-|B| \\ k = i \end{matrix}\right]$ PIE says that the $(n+1) \times (n+1)$-matrix $M = (m_{ij})_{ij}$ with $m_{ij} = \binom{j}{i}$ is invertible & $M^{-1}_{ij} = (-1)^{j-i}\binom{j}{i}$

Example:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

§3 Application: Derangements:

Def: A permutation $\sigma \in \mathfrak{S}_n$ is a derangement if $\sigma(i) \neq i$ $\forall i$ ("σ has no fixed points")

Set $D(n) = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ is a derangement}\}$ TASK: Determine $D(n)$

Example: $D(0) = 0$, $D(1) = 0$, $D(2) = 1$ ($\sigma = 21$), $D(3) = 2$ $(312, 231)$

We interpret $\sigma(i) = i$ as the property e_i for σ's. Set $E = \{e_1, \dots, e_n\}$, $X = \mathfrak{S}_n$

Write $N_{\supseteq T} = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ fixes } T\} = (n-|T|)!$ (σ is identified with $\mathfrak{S}_{[n]\setminus T}$)

$N_{=T} = D(n-|T|)$

So E is homogeneous!

$N_{\supseteq \phi} = \# \mathfrak{S}_n = n!$

So PIE gives ($B=\phi$)

(1) $$n! = \sum_{T \supseteq \phi} N_{=T} \underset{E \text{ homog}}{=} \sum_{k=0}^{n} \binom{n}{k} D(n-k).$$

(2) $$D(n) = \sum_{T\supseteq\phi} (-1)^{|T|} N_{\supseteq T} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k!$$

Note: $D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

We get two recursive expressions for $D(n)$:

(i). $D(n) = n\, D(n-1) + (-1)^n$

(ii). $D(n) = (n-1)\left(D(n-1) + D(n-2)\right)$

for all $n \geq 1$.

(ii) has a combinatorial interpretation:

For each σ derangement, set $j = \sigma(1) \in \{2, \ldots, n-1\}$

- If $\sigma(j) = 1 \Rightarrow \sigma$ switches 1 & j & so σ is a derangement in $\mathfrak{S}_{n-2}$.
- If $\sigma(j) \neq 1 \Rightarrow \tilde{\sigma} = \tau_{1j} \circ \sigma$ fixes 1 & nothing else $\Rightarrow \tilde{\sigma}$ is a derangement in $\mathfrak{S}_{n-1}$.

(i) also has a combinatorial interpretation, but it's MUCH, MUCH HARDER!

Remmel. "A Note on a recursion for the number of derangements"
Europ J. Comb 4 (4) (1983), 371-374.