Lecture X Lingström-Gessel-Viennot's Theorem

(1) **Setup:** G = directed, weighted graph (acyclic, finite)

- V = a set of vertices (finite)
- $E \subset V \times V$ set of edges (directed) (finite) $\quad e = (v_1, v_2) \rightsquigarrow \overset{v_1}{\bullet} \xrightarrow{e} \overset{v_2}{\bullet}$ $\quad v_1 =$ source(e), $v_2 =$ target(e)
- $wt : E \longrightarrow \mathbb{R}$ ($\mathbb{Z}$ or $\mathbb{Q}$, depending on context)

Acyclic means no directed cycles ()

Def: A path from v_1 to v_2 is a sequence of edges $\mathcal{P} = (e_1, e_2, \ldots, e_\ell)$ such that

source$(e_1) = v_1$, Target$(e_\ell) = v_2$

target(e_i) = source(e_{i+1}) for $i = 1, \ldots, \ell-1$

Write $\mathcal{P} : v_1 \dashrightarrow v_2 \qquad wt(\mathcal{P}) := \prod_{i=1}^{\ell} wt(e_i)$ (weight of the path)

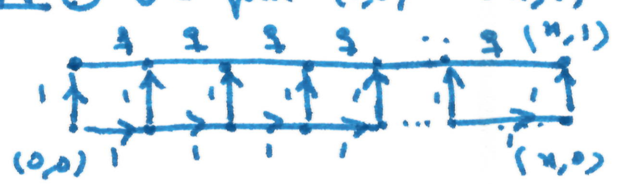
Remark: In general, it will be enough to assume each vertex of G has finite in- & out-deg meaning $\#(\xrightarrow[e]{} v)$ & $\#(\longleftarrow v)$ both finite. For simplicity, we assume G finite for today, but everything works in this larger class.

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two subsets of V.

We define the following matrix $M = (m_{ij})_{1 \leq i, j \leq n}$ via:

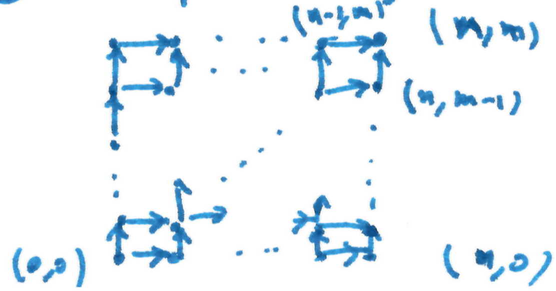
$$m_{ij} = \sum_{P : a_i \dashrightarrow b_j} wt(P) \qquad \forall\, 1 \leq i, j \leq n \qquad (M \text{ depends on } G, A \,\&\, B)$$

Example (1) G = grid $(0,0)$ to $(n,1)$ with edges = E & N.



$$\sum wt(\text{Paths}) = 1 + q + \cdots + q^n = [n+1]_q \qquad \left(\text{weight sum } \sum_{P : (0,0) \to (n,1)} wt(P)\right)$$

(2) G = grid starting at $(0,0)$ & ending at (n,m) with edges = E & N



We look at 2 cases, depending on the weight function:

(I) All weights 1: $wt(e) = 1 \; \forall e$ edge in G.

$\#$ paths $(0,0) \to (n,m) := F_{n,m}$

$$F_{n,m} = F_{n-1,m} + F_{n,m-1}$$

So $F_{n,m} = \binom{n+m}{m}$

(II) Now, if weights are nontrivial:

$$wt\big((j,k) \to (j+1,k)\big) = q^k \qquad \text{(as in Example 1)}$$

Recurrence now becomes $F_{n,m} = q^m F_{n-1,m} + F_{n,m-1}$

This is q-Pascal's identity! $F_{n,m} = \begin{bmatrix} n+m \\ m \end{bmatrix}_q$ [check $F_{(0,0)} = 1$, $F_{(n,1)} = \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q$ by Example 1]

• Next, we state LGV Lemma:

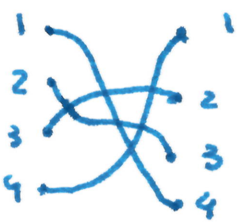
Lemma (LGV): $\det(M) = \sum_{(P_1,\dots,P_n,\sigma)} (-1)^{\sigma}\, wt(P_1)\cdots wt(P_n)$

where $\sigma \in \mathfrak{S}_n$, $P_1 : a_1 \dashrightarrow b_{\sigma(1)}$, $P_2 : a_2 \dashrightarrow b_{\sigma(2)}$, $\vdots$, $P_n : a_n \dashrightarrow b_{\sigma(n)}$ are disjoint paths, ie $\text{Vertices}(P_i) \cap \text{Vertices}(P_j) = \emptyset$ $\forall\, i \neq j$

Q: What is $(-1)^{\sigma}$? $(-1)^{\sigma} := (-1)^{\#\{i<j \mid \sigma(i) > \sigma(j)\}}$

So exponent is $\#\text{inv}(\sigma)$ = number of inversions in σ.

eg



6 inversions so $(-1)^{\sigma} = 1$.

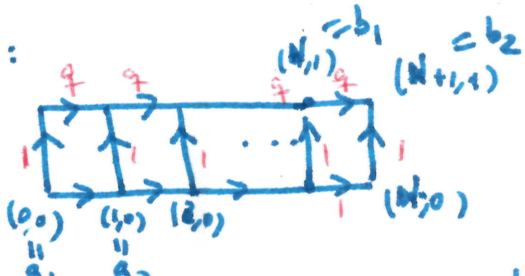
This goes by many names: parity of σ, sign of σ. Write it as a map

$\boxed{\text{sign} : \mathfrak{S}_n \longrightarrow \{\pm 1\}}$ with $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1)\,\text{sign}(\sigma_2)$

(2) Back to old examples:

Example (1) (cont.)

$\boxed{n=2}$



$A = \{a_1, a_2\}$

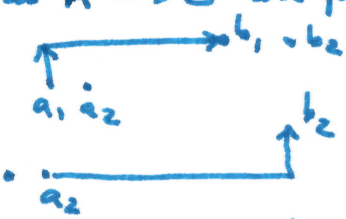
$B = \{b_1, b_2\}$

$$M = \begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & [N+1]_q & [N+2]_q \\ a_2 & [N]_q & [N+1]_q \end{array}$$

Non-intersecting paths $A \to B$ are forced:

$a_1 \longrightarrow b_1$:

$a_2 \longrightarrow b_2$:



All other paths intersect! so $\sigma = id$.

$wt(P_1) = q^N$ & $wt(P_2) = 1$.

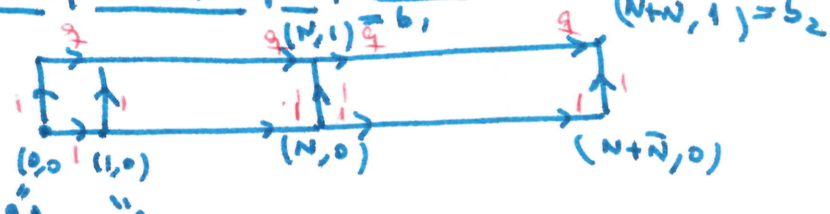
So LGV gives $\det(M) = 1 \cdot q^N \cdot 1$

Can check : $\det(M) = [N+1]_q^2 - [N]_q [N+2]_q$

$$= \frac{(1-q^{N+1})^2 - (1-q^N)(1-q^{N+2})}{(1-q)^2} = \frac{1}{(1-q)^2}\left(\cancel{1} + \cancel{q^{2N+2}} - 2q^{N+1} - (\cancel{1} - q^N - q^{N+2} + \cancel{q^{2N+2}})\right)$$

$$= \frac{q^N(1+q^2-2q)}{(1-q)^2} = q^N \text{ as expected!}$$

Variant of this example : $\boxed{\bar{N}=2}$

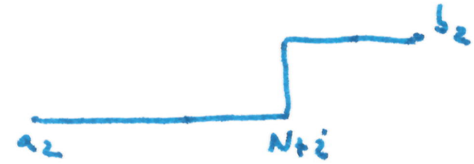


$a_1 = (0,0)$, $a_2 = (1,0)$, $(N,1) = b_1$, $(N+\bar{N},1) = b_2$

$$M = \begin{array}{c} \\ a_1 \\ a_2 \end{array}\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} [N+1]_q & [N+\bar{N}+1]_q \\ [N]_q & [N+\bar{N}]_q \end{bmatrix} \end{array} \Rightarrow \det(M) = ?$$

We count disjoint paths : still have $a_1 \longrightarrow b_1$ & $a_2 \longrightarrow b_2$ for σ & P_1 fixed.

$= P_1 \implies P_2^{(i)}$:



$1 \le i \le \bar{N}$.

A : $= q^N [\bar{N}]_q$. $\quad \left(\sum_{i=1}^{\bar{N}} wt(P_2^{(i)}) = [\bar{N}]_q \right)$.

Proof of LGV :

We introduce the following notation :

Notation : For $\sigma \in \mathfrak{S}_n$:

$$\mathcal{P}(A;B;\sigma) = \left\{ (P_1,\dots,P_n) \;\middle|\; \begin{array}{l} P_1 : a_1 \dashrightarrow b_{\sigma(1)} \\ P_2 : a_2 \dashrightarrow b_{\sigma(2)} \\ \vdots \\ P_n : a_n \dashrightarrow b_{\sigma(n)} \end{array} \right\} \text{ (all paths)}$$

$$\cup$$

$$\mathcal{P}^\circ(A;B;\sigma) = \left\{ (P_1,\dots,P_n) \;\middle|\; \begin{array}{l} P_i : a_i \dashrightarrow b_{\sigma(i)} \quad (1 \le i \le n) \\ \mathrm{Vertex}(P_i) \cap \mathrm{Vertex}(P_j) = \emptyset \quad \forall i \ne j \end{array} \right\}$$

We have a formula for $\det(M)$ involving all permutations :

$$\det(M) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \, m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}$$

Now, we use the definition of m_{ij}

$m_{1,\sigma(1)}$ = weighted sum of paths $a_1 \dashrightarrow b_{\sigma(1)}$

So
$$\det(M) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} \sum_{(P_1, \ldots P_n) \in \mathcal{P}(A,B;\sigma)} wt(P_1) \cdots wt(P_n)$$

To show:
$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} \sum_{\underline{P} = (P_1, \ldots P_n) \in \mathcal{P}(A,B,\sigma)} wt(P_1) \cdots wt(P_n) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} \sum_{(P_1, \ldots P_n) \in \mathcal{P}^0(A,B,\sigma)} wt(P_1) \cdots wt(P_n)$$

Equivalently:

To show:
$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} \sum_{\underline{P} \in \mathcal{P}^{\times}(A,B,\sigma)} wt(P_1) \cdots wt(P_n) = 0$$
where $\underline{P} = (P_1, \ldots P_n)$ & $\mathcal{P}^{\times}(A,B,\sigma) = \mathcal{P}_{(A,B,\sigma)} \setminus \mathcal{P}^0_{(A,B,\sigma)}$ (crossing paths)

We prove it bijectively:

Claim: There is a bijection:
$$\varphi : \bigcup_{\sigma \in \mathfrak{S}_n} \mathcal{P}^{\times}(A,B,\sigma) \longrightarrow \bigcup_{\tau \in \mathfrak{S}_n} \mathcal{P}^{\times}(A,B,\tau)$$
$$(P_1, \ldots P_n) \longmapsto \varphi(\underline{P}) = (P_1', \ldots, P_n') \in \mathcal{P}(A,B,\sigma')$$

satisfying 3 properties:

(1) $wt(P_1) \cdots wt(P_n) = wt(P_1') \cdots wt(P_n')$

(2) $(-1)^{\sigma'} = -(-1)^{\sigma}$

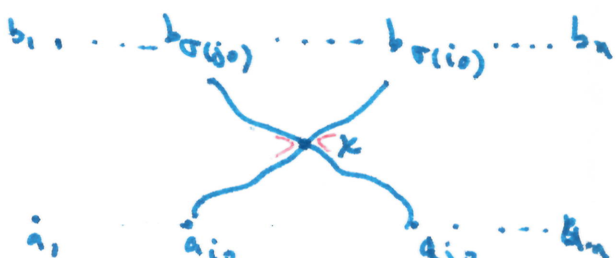
(3) $\varphi^2 = id$ (involution)

[Terms will cancel out]

Proof of claim:

Given $(P_1, \ldots, P_n) \in \mathcal{P}^{\times}(A,B;\sigma)$,

- Let i_0 be the smallest index st P_{i_0} crosses another path.
- Let x be the first (time) vertex on P_{i_0} which also appears in another path.
- Let j_0 be the smallest index such that $P_{i_0} \cap P_{j_0} \ni x$



Then: $\varphi(P_1 \ldots P_n) = (P_1', \ldots, P_n')$ is defined to be: $P_k' = P_k \quad \forall k \neq i_0, j_0$

$P_{i_0}' = a_{i_0} \xrightarrow{P_{i_0}} x \xrightarrow{P_{j_0}} b_{\sigma(j_0)}$

$P_{j_0}' = a_{j_0} \longrightarrow x \xrightarrow{P_{i_0}} b_{\sigma(i_0)}$

(2) $\sigma' = (i_0, j_0) \circ \sigma$ so $\operatorname{sign} \sigma' = -\operatorname{sign}(\sigma)$ & property (2) holds

- Edges in $\cup$ paths don't change so Property (1) is clear
- Involution is also clear $\blacksquare$

Remark: This idea is very useful for checking that a signed sum cancels:
In general, find an involution (φ) on the indexing set of the sum so that we identify pairs of summands that have different signs $[(-1)^{\sigma}$ & $(-1)^{\varphi(\sigma)} = -(-1)^{\sigma}]$.