Lecture XI: Applications of Lindström-Gessel-Viennot's Lemma

Recall: $G=(V,E)$ directed, finite acyclic graph with $wt: E \to \mathbb{R}$ weight function, $A=\{a_1,\ldots,a_n\}$, $B=\{b_1,\ldots,b_n\}$ & $M=(m_{ij})_{1\le i,j\le n}$, $m_{ij}=\sum_{\substack{P: a_i \dashrightarrow b_j \\ \text{path}}} wt(P)$ $\forall i,j$, where

$$w(P)=\prod_{e\in P} wt(e)$$

Lemma (LGV):
$$\det(M)=\sum_{\sigma\in \mathfrak{S}_n}(-1)^{\sigma}\sum_{\underline{P}\in \mathcal{P}^{\circ}(A,B,\sigma)} wt(P_1)\cdots wt(P_n)$$

with $\underline{P}=(P_1,\ldots,P_n)$ & $\mathcal{P}^{\circ}(A,B,\sigma)=\{P_i: a_i \dashrightarrow b_{\sigma(i)}\ \ i=1,\ldots,n \mid V(P_i)\cap V(P_j)=\emptyset\ \forall i\neq j\}\subseteq \mathcal{P}(A,B,\sigma)$

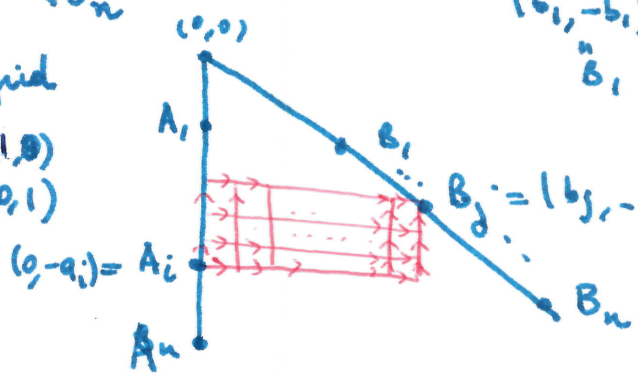
TODAY: Applications of this lemma → computing $\det(M)$ from path enumeration
→ path enumeration using $\det(M)$.

§1 Binomial determinants:

Reference: Gessel, Ira & Viennot, Gérard: "Binomial determinants, paths & hook-length formulae". Adv in Math. 58 (1985), no 3, 300-321.

Fix $0\le a_1<\cdots<a_n$, $0\le b_1<\cdots<b_n$ in $\mathbb{Z}$ & define points $\underset{A_1}{(0,-a_1)},\ldots,\underset{A_n}{(0,-a_n)}$, $\underset{B_1}{(b_1,-b_1)},\ldots,\underset{B_n}{(b_n,-b_n)} \in \mathbb{Z}^2_{\le 0}$.

Plot them in grid with steps $E=(1,0)$, $N=(0,1)$



- If $a_i\ge b_j$ we can place a rectangle of size $b_j\times(a_i-b_j)$ with lower left corner A_i & upper right corner B_j.

[If $a_i<b_j$, the rectangle has the same size but A_i is upper left corner & B_j is lower right]

Define $M=\left(m_{ij}=\binom{a_i}{b_j}\right)_{1\le i,j\le n}$ $\qquad \binom{a_i}{b_j}=\binom{b_j+(a_i-b_j)}{b_j}$ path count $A_i\to B_j$

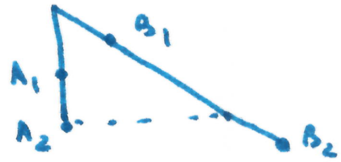
Goal: compute $\det(M)$.

Q: What is $\mathcal{P}^{\circ}(A,B,\sigma)$? What is $\mathcal{P}(A,B,\sigma)$?

- If $a_1<b_1$, then A_1 is above all B_i's so 1^{st} row of $M=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ & $\det M=0$.

$\Rightarrow$ Assume $a_1\ge b_1$, i.e. can build a path from A_1 to B_1.

If $a_2 < b_2$ $\Rightarrow$



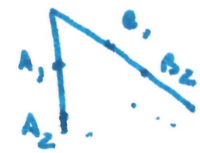
$$\Rightarrow M = \left(\begin{array}{c|ccc} \binom{a_1}{b_1} & 0 & \cdots & 0 \\ \hline \binom{a_2}{b_1} & 0 & \cdots & 0 \\ \vdots & & & \\ \binom{a_n}{b_1} & & & \end{array}\right) \begin{array}{l} \to \binom{a_1}{b_j}\ b_j > a_1 \\ \to \binom{a_2}{b_j}\ b_j > a_2 \\ \\ \\ \end{array}$$

So $\det M = 0$

<u>Inductively</u>: If $a_1 \geq b_1, \ldots, a_k \geq b_k$ <u>but</u> $a_{k+1} < b_{k+1}$ $\Rightarrow$ $\det M = 0$

<u>Conclusion</u>: We need only look at the case $\boxed{a_j \geq b_j \quad \forall j = 1, \cdots n}$

<u>Claim</u>: To have non-crossing paths, the only option is $\sigma = id$.



<u>Proof</u>: If $A_1 \xrightarrow{P_1} B_j$ for $j \neq 1$, pick i with $A_i \xrightarrow{P_i} B_1$ $(i > 1)$

Then the paths P_1 & P_i are forced to intersect each other because:

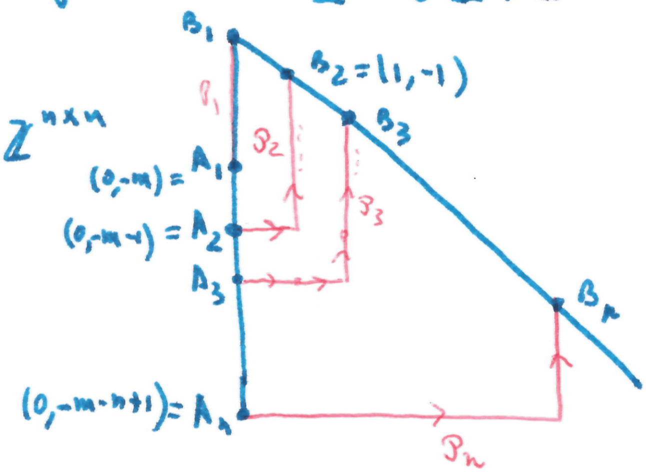
$$ht(\underbrace{source(P_i)}_{A_i}) < ht(\underbrace{source(P_1)}_{A_1}) \quad \text{but} \quad ht(\underbrace{target(P_i)}_{B_1}) > ht(\underbrace{target\ P_1}_{B_j})$$

& path $\xrightarrow{ht} \mathbb{R}$ is a continuous function. □.
$(x, y) \longmapsto y$

<u>Claim 3</u>: $\mathcal{P}^0(A, B, id) \neq \emptyset$ when $a_j \geq b_j$ $\forall j = 1 \cdots n$ & so $\boxed{\det(M) = |\underbrace{\mathcal{P}^0(A, B, id)}_{>0}|}$

<u>Example 1</u>: $(a_i), (b_i)$ defined by $a_i = m + i - 1$ so $\underline{a}$: $m < m-1 < \cdots < m+n-1$
$b_j = j - 1$ $\underline{b}$: $0 \leq 1 \leq \cdots < n-1$

$$M = \begin{pmatrix} \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{n-1} \\ \binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{n-1} \\ \vdots & & & \\ \binom{m+n-1}{0} & \binom{m+n-1}{1} & \cdots & \binom{m+n-1}{n-1} \end{pmatrix} \in \mathbb{Z}^{n \times n}$$



<u>Claim</u>: $|\mathcal{P}^0(A, B, id)| = 1$

<u>Proof</u>: . P_1 has only one option

. Once P_1 is fixed, only option for P_2 is

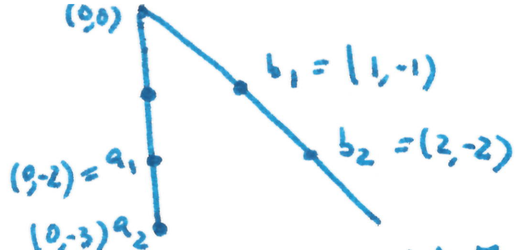
. Once P_2 ———, ——————— P_3

□

Since $wt(e) = 1$ $\forall e$ edge, we conclude by LGV that $\boxed{\det M = 1}$

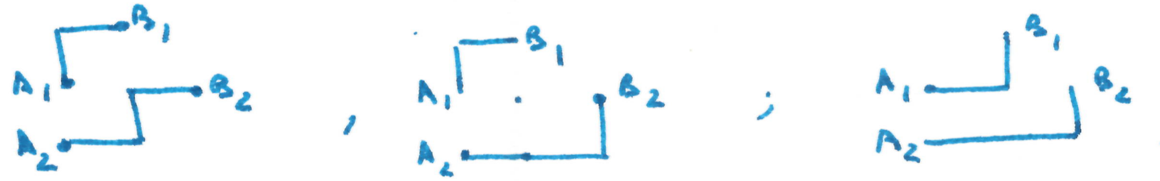
Example 2: $n=2$ $\quad a_1 = 2, \quad a_2 = 3$

$b_1 = 1, \quad b_2 = 2$



$$\Pi = \begin{bmatrix} \binom{2}{1} & \binom{2}{2} \\ \binom{3}{1} & \binom{3}{2} \end{bmatrix}$$

In this case $\sigma = id$ is only relevant case, so we can use $\det \Pi = \det \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} = 3$ to conclude that $|\mathcal{P}^0(A, B, id)| = 3$.



Once the paths are computed, we can change the weights to compute some other determinants, e.g. involving q-numbers.

§2 Hankel matrices:

. Fix a sequence of real numbers $\underline{Q} = (Q_0, Q_1, Q_2, \dots)$.

Def: The Hankel matrices of $\underline{Q}$ are sequences of pairs of matrices: $(H_n, H_n^{(1)})$ with

$$H_n = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_n \\ Q_1 & Q_2 & \cdots & Q_{n+1} \\ \vdots & & & \vdots \\ Q_n & Q_{n+1} & & Q_{2n} \end{bmatrix} = (Q_{i+j})_{i,j=0}^{n}$$

&

$$H_n^{(1)} = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_{n+1} \\ Q_2 & Q_3 & \cdots & Q_{n+2} \\ \vdots & & & \\ Q_{n+1} & Q_{n+2} & \cdots & Q_{2n+1} \end{bmatrix} = (Q_{i+j+1})_{i,j=0}^{n}$$

From $\underline{Q}$ we construct the sequence $\underline{D} = (\det H_0, \det H_0^{(1)}, \det H_1, \det H_1^{(1)}, \dots)$.

Claim: If $0 \notin \underline{D}$, we can recover $\underline{Q}$ uniquely from this determinant sequence.

Proof: $\det H_0 = Q_0$, $\det H_0^{(1)} = Q_1$,

$$\begin{cases} \det H_1 = \det \begin{pmatrix} Q_0 & Q_1 \\ Q_1 & Q_2 \end{pmatrix} = Q_0 Q_2 - Q_1^2 \overset{Q_0 \neq 0}{\Longrightarrow} \text{recover } Q_2 \\ \det H_1^{1} = \det \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{pmatrix} = Q_1 Q_3 - Q_2^2 \underset{\substack{Q_1 \neq 0 \\ \text{know } Q_2}}{\Longrightarrow} \text{recover } Q_3 \end{cases}$$

Continue in this way by induction.

Question: Which sequence $\underline{Q}$ yields $\det H_n = \det H_n^{(1)} = 1$ for all n?

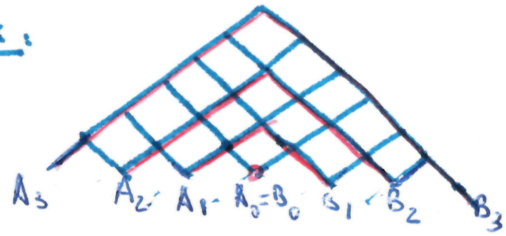
We give an answer with LSV.

Consider the lattice graph above the x-axis with diagonal steps $(1,1)$ & $(1,-1)$

all weights $= 1$.

Fix $a_i = (-2i, 0)$, $b_j = (2j, 0)$ $i, j = 0, \dots n$

Ex.:



Only option to have non-intersecting paths is $\sigma = id$

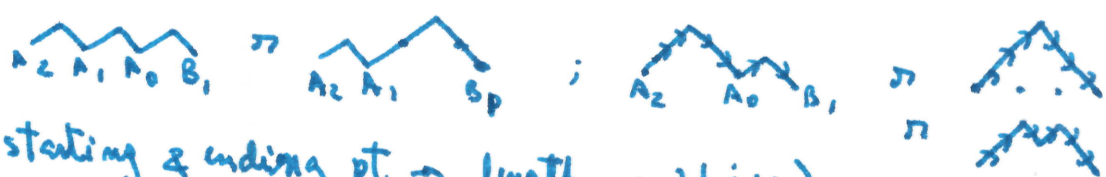
Claim: A_{i+j} = sum of weighted paths from A_i to B_j satisfies:

$\det H_n = 1$ by LGV

Eg: $A_0 \to B_0$  (length $= 2$)

$A_2 \to B_0$  (length $= 4$)

$A_1 \to B_1$  (length $= 4$)

$A_2 \to B_1$  (length $= 6$)

- Independent of starting & ending pt $\Rightarrow$ length $= 2(i+j)$.
- So Q_k = number of paths of length $2k$ in the grid above.

$\det H_n^{(1)} = \det(Q_{i+j+1}) = \det \begin{array}{c|c} & B_1 \quad B_2 \quad \cdots \quad B_{n+1} \\ \begin{array}{c} A_0 \\ A_1 \\ \vdots \\ A_n \end{array} & \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \end{array}$ again, only one option for σ

$A_0 \to B_1$

$A_1 \to B_2$

$\vdots$

So $\det H_n^{(1)} = 1$. by LGV.

Claim 2: $Q_k = k^{th}$ Catalan number $\left(\begin{array}{l} C_0 = 1 \\ C_1 = 1 \\ C_2 = 2 \end{array} \right.$ $C_3 = 5 \rightsquigarrow \boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$)

Pf.: Future lecture