Lecture XII: Counting number partitions

Last Time: Use LGV to compute det(M) from non-intersecting paths.

Next, we want to use det(M) to find the exact number of paths.

Main example = ~~Plake~~ partitions. (next time) **Before:** study partitions (of integers)

§1 (Number) partitions:

Def: A *partition* of $n \in \mathbb{Z}_{\geq 1}$ is a sequence $(\lambda_1, \lambda_2, \dots)$ s.t $\sum_{i=1}^{\infty} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$.

Write $(\lambda_1, \dots, \lambda_k)$ for the non-zero parts & denote this by $\boxed{\lambda \vdash n}$ with $|\lambda| = n$

(Here $|\lambda| = \sum_{i=1}^{k} |\lambda_i|$) To emphasis k, we can refer to λ as a *k-partition*.

Notes: (1) Write $\mathrm{Par}(n) = \{\text{partitions of } n\} \supseteq \mathrm{Par}(n;k) = \{k\text{-partitions of } n\}$

Set $P(n) := |\mathrm{Par}(n)|$ & $P(n;k) := |\mathrm{Par}(n;k)|$.

Example: $P(0) = 1$, $P(0,k) = 0$ for $k > 1$; $P(0,1) = 1$.

(2) k-Partitions are k-compositions without an order (ie rearrange so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$)

(3) Hard to count because they aren't ordered

Example: For $n=5$ we have $P(5) = 7$: 5, 41, 32, 311, 221, 2111, 11111

(If ordered: $311 \neq 131 \neq 113$.)

Obs: • The number of ordered k-partitions of n is $\binom{n-1}{k-1}$ = # k-comp of n.

• There are 2^{n-1} ordered partitions of n (= # comp of n)

Q1: Recurrences? Write $\mathrm{Par}(n, \leq k) = \{\text{partitions of } n \text{ with } \underline{\text{at most}}\ k \text{ parts}\}$

$= \bigsqcup_{s=0}^{k} \mathrm{Par}(n,s)$.

& set $P(n, \leq k) := |\mathrm{Par}(n, \leq k)|$

We get ~~the~~ recurrence pretty easily:

Lemma: $P(n,k) = P(n-k, \leq k) = P(n-k;1) + P(n-k;2) + \dots + P(n-k,k)$.

Proof Define a bijection $\phi : \mathrm{Par}(n;k) \longrightarrow \mathrm{Par}(n-k; \leq k)$

$(\lambda_1, \dots, \lambda_k) \longmapsto (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_k - 1)$

We have $\leq k$ parts on the right since we may have a tail of 0's. $\square$

Corollary: $P(n,k) = P(n-1,k-1) + P(n-k;k)$

$ \overset{\|}{P(n-k;\le k-1)}$

- Table for small values of $P(n,k)$ & $P(n)$ using recurrence:

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	1	1				
4	0	1	2	1	1			
5	0	1	2	2	1	1		
6	0	1	3	3	2	1	1	
7	0	1	3	4	3	2	1	1

n parts for $n = (\underbrace{1,1,1,\ldots,1}_{n \text{ times}})$

$(n-1)$ parts for $n = (2,\underbrace{1,1,\ldots,1}_{n-2})$

1 part for $n = (n)$

Eg $P(7,3) = P(6,2) + P(4,3)$
$P(6,2) = P(5,1) + P(4,2)$

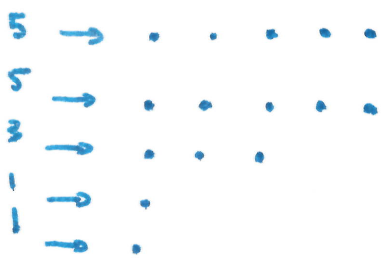
n	0	1	2	3	4	5	6	7
$P(n)$	1	1	2	3	5	7	11	15

Q2: How to represent partitions?

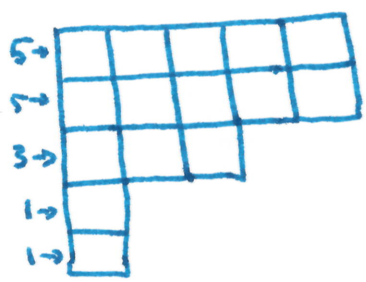
- For compositions, dots & bars were a helpful schematic representation.
- For partitions, we use Ferrers diagrams or Young diagrams

Ex: $1+1+3+5+5 = 15 \rightsquigarrow (5,5,3,1,1) \vdash 15$.

Ferrer diagram vs. Young diagram



draw aligned dots in rows



stack boxes in rows

Note: In advanced Combinatorics, often fill Young diagrams with numbers in boxes (SYT, SSYT)

Q3: How many partitions? Nobody knows.

Thm (Hardy-Ramanujan, 1918) $P(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\left[\pi\sqrt{\frac{2n}{3}}\right]}$ as $n \to \infty$

§2. How to count partitions

Approach to counting: define an involution map:

Def: Given a partition $\lambda \vdash n$, the conjugate partition λ^* of n is defined by reflecting the Ferrer or Young diagram of λ along the main diagonal $y=-x$.

Ex above: $\lambda =$

$= \lambda^* = (5,4,3,2,2)$

Easy exercise: λ^* defines a partition (of n). [flip rows & columns of Ferrers diagram & show $\#\text{col}_1 \geq \#\text{col}_2 \geq \cdots$ in λ.]

Follows from the observation:

Obs: • $\lambda_i^* = \{ j : \lambda_j \geq i \}$

• $\#\text{parts}(\lambda^*) =$ highest summand λ_1 of λ. $\Longrightarrow$ can incorporate this into our counting approach.

Def: $\mathcal{P}ar(n;k;m) = \{$ partitions of n with k parts & $\lambda_1 = m \}$ $\rightsquigarrow$ size $= P(n;k;m)$

$\mathcal{P}ar(n;k;\leq m) = \{$ ———————— & $\lambda_1 \leq m \}$ $\rightsquigarrow$ size $= P(n;k;\leq m)$

$\mathcal{P}ar(n, \leq k, m) = \{$ ———— $\leq k$ ———— & $\lambda_1 = m \}$ $\rightsquigarrow$ size $= P(n;\leq k;m)$

$\mathcal{P}ar(n;\leq k;\leq m) = \{$ ———— $\leq k$ ———— & $\lambda_1 \leq m \}$ $\rightsquigarrow$ size $= P(n;\leq k;\leq m)$

$\mathcal{P}ar(\,;\leq k;\leq m) = \{$ partitions with $\leq k$ parts & $\lambda_1 \leq m \}$ $\rightsquigarrow$ size $P(\,;\leq k;\leq m)$

Example: $k=3$, $m=2$:

$\mathcal{P}ar(\,;\leq 3;\leq 2) =$

222	22	2	$\emptyset$
221	21	1	
211	11		
111			

so $P(\,;\leq 3;\leq 2) = 10$.

• Involution exchanges the role of k & m. Thus:

Prop:
$$P(n;k;m) = P(n;m;k)$$
$$P(n;k;\leq m) = P(n;\leq m;k)$$
$$P(\,;\leq k;\leq m) = P(\leq m;\leq k)$$

Special partition: $\lambda_1 > \lambda_2 > \cdots\cdots$ Write the sets as $\mathcal{P}ar_d(n)$; $\mathcal{P}ar_d(n;k;m)$ etc., with cardinalities $P_d(n)$, $P_d(n;k;m)$, etc. ["d" for "distinct elements"]

Lemma:
$$P_d(n;k;\leq m) = P\left(n - \binom{k+1}{2};\leq k;\leq m-k\right)$$
$$P_d(\,;k;\leq m) = P(\,;\leq k;\leq m-k)$$

Proof: Define 2 bijections
$$\phi : \mathcal{P}ar_d(n;k;\leq m) \longrightarrow \mathcal{P}ar\left(n - \binom{k+1}{2};\leq k;\leq m-k\right)$$
$$\phi : \mathcal{P}ar_d(\,;\underset{\cap}{k};\leq m) \longrightarrow \mathcal{P}ar(\,;\underset{\cap}{\leq k};\leq m-k)$$

via $\phi(\lambda_1,\dots,\lambda_k) = (\lambda_1 - k, \lambda_2 - (k-1), \dots, \lambda_k - 1)$, omitting the potential tail of 0's.

- $|\phi(\lambda_1,\dots,\lambda_k)| = |\lambda| - \sum_{i=1}^{k} i = |\lambda| - \frac{k(k+1)}{2} = |\lambda| - \binom{k+1}{2} = n - \binom{k+1}{2}$
- # parts of $\phi(\lambda_1,\dots,\lambda_k) \leq k$.
- $\phi(\lambda_1,\dots,\lambda_k)_1 = \lambda_1 - k \leq m - k$.
- The map is a bijection by construction & restriction of ϕ from $\mathrm{Par}_d(;k;\leq m)$ has image $\mathrm{Par}(n - \binom{k+1}{2}; \leq k; \leq m-k)$. □

§3 Lattice Paths:

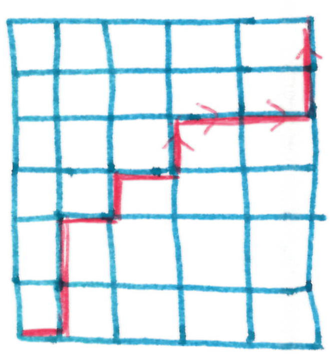
Recall: $L(m,n)$ = lattice paths using $E=(1,0)$ & $N=(0,1)$ from $(0,0)$ to (m,n)

$|L(m,n)| = \binom{m+n}{m}$.

Prop: $\exists$ bijection $L(m,n) \longrightarrow \mathrm{Par}(;\leq n;\leq m)$ so $p(;\leq k;\leq m) = \binom{m+k}{m}$

Proof: Interpret "upper half of the grid  $(0,0)$ (m,n) defined by a path in $L(m,n)$ as a Ferrers diagram. □

Example:



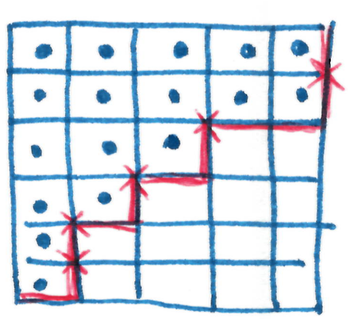
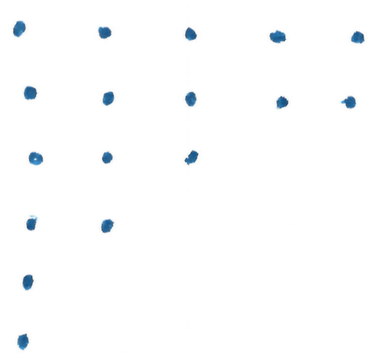
$m=5, n=6$

$\longrightarrow \lambda = (5,5,3,2,1,1) \vdash 17$

$p(;\leq 6;\leq 5) = \binom{11}{6} = 462$

(Next Time: Use LGV to count partitions.)

For Ferrers diagram representation: draw a • within each box.

 $\mapsto$ 

size of each strip = end-points of N step (marked with $\times$)