Lecture XIII: Applications of Lindström-Gessel-Viennot's Lemma

Last time: defined partitions $\lambda \vdash n$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ with $|\lambda| = \sum_i \lambda_i = n$

length$(\lambda) = \{k \mid \lambda_k > 0 \;\&\; \lambda_{k+1} = 0\}$

$\mathcal{P}ar(n; k; m) = \{\lambda \vdash n \text{ length}(\lambda) = k,\ \lambda_1 = m\}$ $\rightsquigarrow |\mathcal{P}ar(n,k,m)| = P(n,k,m)$

Also added $(\leq k; \leq m)$ combinations

Prop: $\mathcal{P}ar(\,; \leq k; \leq m) = \{$partitions λ of length$(\lambda) \leq k$ & $\lambda_1 \leq m\}$ $\xleftrightarrow{1-1} L(k,m)$

so $P(\,; \leq k \leq m) = \binom{m+k}{m}$

TODAY: Plane partitions

§1 Plane partitions

Def: A plane partition of $n \in \mathbb{Z}_{\geq 1}$ is an array of integers $\underline{\lambda} = (\lambda_{ij})_{i,j}$ with

$$= \begin{matrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots \\ \lambda_{21} & \lambda_{22} & \cdots & \\ \vdots & \vdots & & \end{matrix}$$

- $\lambda_{i,j} \geq 1$ & $|\underline{\lambda}| = \sum \lambda_{ij} = n$
- Every row $\lambda_{i1} \geq \lambda_{i2} \geq \cdots$ & every column $\lambda_{1j} \geq \lambda_{2j} \geq \cdots$ (so they are weakly monotone)

Obs: $\lambda_{ik} \geq \lambda_{jl}$ for all $i \leq j,\ k \leq l$

Example: $n = 15$

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{matrix}$$

Set $\mathcal{PP}ar(n; \leq r; \leq s; \leq t) = \{$plane partitions of n with at most r rows & s columns & $\lambda_{11} \leq t\}$ (largest entry)

$\mathcal{PP}ar(\,; \leq r \leq s \leq t) = \{$ ______ ______ $\}$

$PP(n; \leq r, \leq s \leq t) = |\mathcal{PP}ar(n; \leq r; \leq s; \leq t)|$ & $PP(\,; \leq r; \leq s; \leq t) = |\mathcal{PP}ar(\,; \leq r; \leq s; \leq t)|$

Example
- $PP(0; \leq r; \leq s; \leq t) = 1$
- $PP(\,; \leq 2; \leq 2; \leq 2) = 20 = 1 \cdot 4 + 4 \cdot 3 + 1 \cdot 4$

$n=0$: $\begin{matrix}0&0\\0&0\end{matrix}$ $n=1$: $\begin{matrix}1&0\\0&0\end{matrix}$ $n=2$: $\begin{matrix}1&1\\0&0\end{matrix}$ $\begin{matrix}1&0\\1&0\end{matrix}$ $\begin{matrix}2&0\\0&0\end{matrix}$ $n=3$: $\begin{matrix}1&1\\1&0\end{matrix}$ $\begin{matrix}2&1\\0&0\end{matrix}$ $\begin{matrix}2&0\\1&0\end{matrix}$ $n=4$: $\begin{matrix}1&1\\1&1\end{matrix}$ $\begin{matrix}2&2\\0&0\end{matrix}$ $\begin{matrix}2&0\\2&0\end{matrix}$ $\begin{matrix}2&1\\1&0\end{matrix}$

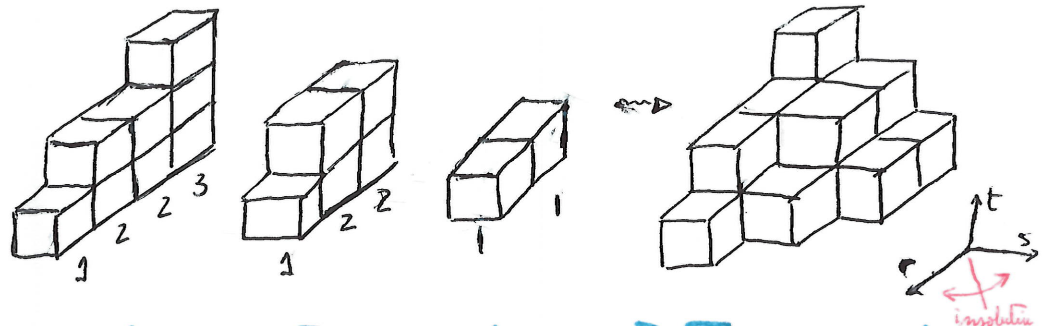
$n=8$: $\begin{matrix}2&2\\2&2\end{matrix}$ $n=7$: $\begin{matrix}2&2\\2&1\end{matrix}$ $n=6$: $\begin{matrix}2&2\\1&1\end{matrix}$ $\begin{matrix}2&1\\2&1\end{matrix}$ $\begin{matrix}2&2\\2&0\end{matrix}$ $n=5$: $\begin{matrix}2&1\\1&1\end{matrix}$ $\begin{matrix}2&2\\1&0\end{matrix}$ $\begin{matrix}2&1\\2&0\end{matrix}$

$$\text{gf} = \sum_{n \geq 0} PP(n; \leq r; \leq s; \leq t)\, q^n = 1 + q + 3q^2 + 3q^3 + 4q^4 + 3q^5 + 3q^6 + q^7 + q^8.$$

Q: How to visualize plane partitions?

$\underline{\lambda} = (\lambda_{ij}) \rightsquigarrow$ stack λ_{ij} unit cubes into an $r \times s \times t$ box.

Example:
$$\begin{matrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{matrix}$$



Advantage: We have a natural involution on plane partitions induced by reflection along $r \leftrightarrow s$. (same as for ordinary partitions)

$$\underline{\lambda} \longrightarrow \underline{\lambda}^* = \lambda^t \quad \text{(transpose the "matrix } \lambda\text{")} = \text{conjugate partition}$$

Example $\lambda = \begin{matrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{matrix} \rightsquigarrow \lambda^* = \begin{matrix} 3 & 2 & 2 & 1 \\ 2 & 2 & 1 & \\ 1 & 1 & & \end{matrix}$

Lemma $\mathcal{P}ar(n; \leq r; \leq s; \leq t) \longrightarrow \mathcal{P}ar(n; \leq s; \leq r; \leq t)$ is an involution
$$\lambda \longmapsto \lambda^*$$

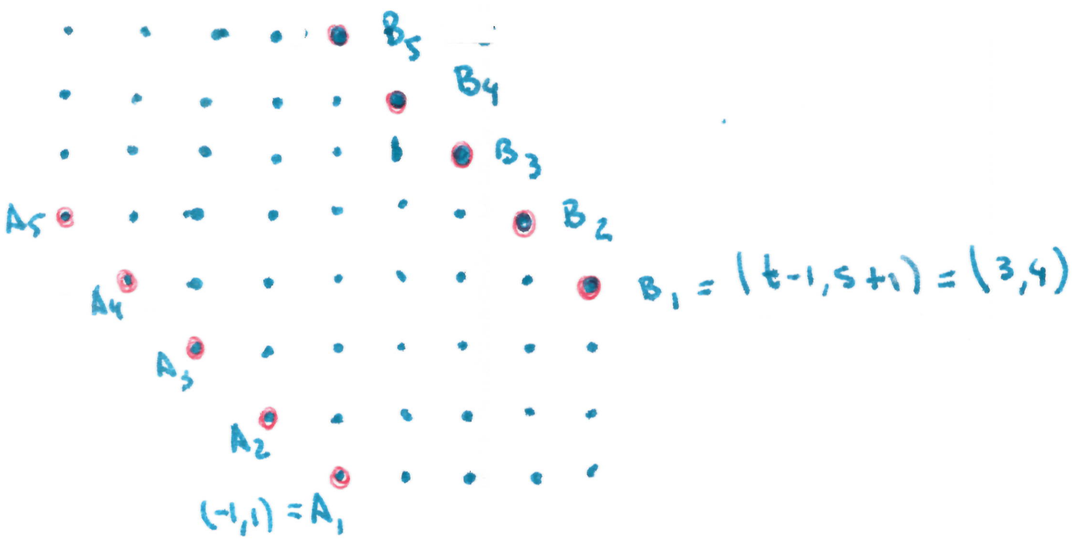
Thus $pp(; \leq r; \leq s; \leq t) = pp(; \leq s; \leq r; \leq t)$

§1 Counting ~~pl~~ plane partition

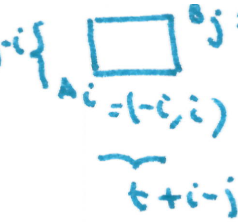
GOAL: Use LGV Lemma to compute $pp(\leq r; \leq s; \leq t)$ $[r, s, t \geq 0]$

- Use usual lattice graph G with steps $N = (0,1)$ & $E = (1,0)$ (all weights $= 1$)
- Set $\mathcal{A} = \{A_1, \ldots, A_r\}$ $\quad A_i = (-i; i)$ for all $i = 1, \ldots, r$
  $\mathcal{B} = \{B_1, \ldots, B_r\}$ $\quad B_j = (t-j, s+j)$ for all $j = 1, \ldots, r$

Example: $r = 5$, $s = 3$, $t = 4$.



• Path matrix $M = (m_{ij}) = \big(\text{Path}(A_i \to B_j)\big)$



$$= \binom{t+i-j+s+j-i}{s+j-i}$$

$$= \binom{t+s}{s+j-i}$$

• **Claim 1:** $\mathcal{P}^\circ(A,B,\sigma) \neq \emptyset \iff \sigma = id$ (use ht function)

[& $i \le s+i$ & $t-i \ge -i$ $\forall i$]

Claim 2 $\mathcal{P} = (P_1, \dots, P_r) \in \mathcal{P}(A,B,id)$. Then $P_i \longleftrightarrow (\lambda_{i1}, \lambda_{i2}, \dots \lambda_{ik}) = \lambda(P_i)$

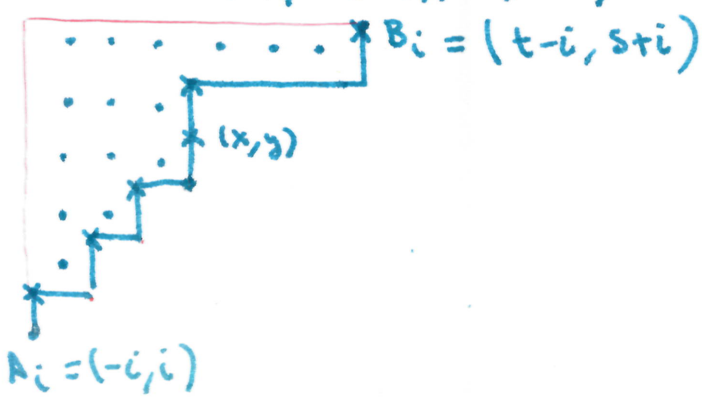
$\cap\, L(s,t) \simeq L(t,s)$ with $\lambda_{i1} \le t$, $k \le s$

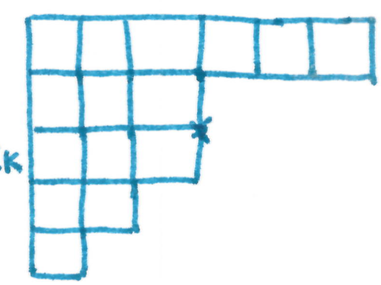
How? Look at Ferrers diagram above the path in the grid

• If $k < s$, fill number partition $\lambda(P_i)$ with a tail of 0's, so we can assume $\lambda(P_i) = (\lambda_{i1}, \dots, \lambda_{is})$

Thm: $\lambda(P_1), \dots \lambda(P_r)$ form a plane partition if and only if $\mathcal{P} \in \mathcal{P}^\circ(A,B,id)$

Proof: Write $\lambda(P_i) = (\lambda_{i1}, \dots, \lambda_{is})$





$k = 3 = 6-4+1$

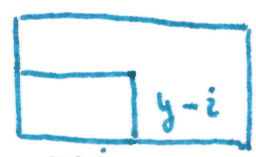
$(s=6,\ t = 7,\ x+i = 3)$

$y - i = 4$

• Mark points at the end of each N step (6 × in picture above)

They correspond to leftmost points in Ferrers diagram.

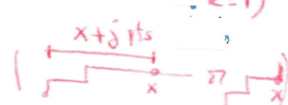
• $\boxed{\lambda_{ik} = x+i}$ & $\boxed{k = s+i+1-y}$ (length of grid:



$= \#$ rows of $\lambda(P_i)$ = position $+1$. (if $y=s$, $k=1$)

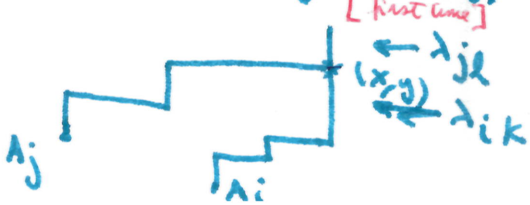
We show $(\Rightarrow)$ by contradiction. The converse is almost identical.

Pick $i<j$; $P_i \cap P_j \ni (x,y)$ [first time] if and only if $\lambda_{j\ell} \ge x+j$



$\ell = s+j+1-(y+1) = s+j-y$

$\Big[\text{ht} = \boxed{\begin{smallmatrix} y+1 \\ y \end{smallmatrix}}\Big\}\, s+j \Big]$



Since (x,y) is end pt for N in $\mathcal{P}_i$ $\quad \lambda_{ik} = x+i$ & $k = s+i+1-y$

so $s-y = k-i-1$.

So $\ell = s-y+j = k-i-1+j = k+\underbrace{j-(i+1)}_{\geq 0 \ (i<j)} \Rightarrow \boxed{\ell \geq k}$

$\lambda_{j\ell} \geq x+j > x+i = \lambda_{ik}$ but $\lambda_{ik} \geq \lambda_{j\ell}$ if λ is a plane partition

(i,k) $\bullet j,\ell$

Conclude: $\mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset \Rightarrow \lambda$ is not a plane partition. $\square$

Corollary: $PP(;\leq r;\leq s;\leq t) = \det \begin{pmatrix} t+s \\ s+j-1 \end{pmatrix}_{1\leq i,j\leq r}$

Pf/ Thm + LGV & definition of Π. $\square$

Example: $r=s=t=2$ $\quad PP(;\leq 2;\leq 2;\leq 2) = \det \begin{pmatrix} \binom{4}{2} & \binom{4}{3} \\ \binom{4}{1} & \binom{4}{2} \end{pmatrix} = \det \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$

$= 36-16 = 20.$