Lecture XIV Inversions for (multiset)-permutations & multinomial q-coefficients

Recall: Given $\sigma \in \mathfrak{S}_n$ permutation, an *inversion* of σ is a pair $(\sigma(i), \sigma(j))$ where $i<j$ & $\sigma(i) > \sigma(j)$. Write $\mathrm{inv}(\sigma) :=$ total # of inversions of σ

Thm (MacMahon): $\displaystyle\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!$

Proof: Bijection $I: \mathfrak{S}_n \longrightarrow T_n = [0,n-1]\times[0,n-2]\times\cdots\times[0,n-i]\times\cdots\times[0,0]$

$\sigma \longmapsto (a_1,\ldots,a_n) =$ inversion table for σ — $[0,n-i]$ encodes all monomials q^{a_i} in $[n-i]_q$.

satisfying $\mathrm{inv}(\sigma) = a_1+\cdots+a_n$. □

$a_i :=$ # of elements to the left of $(n-i)$ in σ that are larger than $(n-i)$.

Example: $\sigma = 21354 \in \mathfrak{S}_5 \xrightarrow{I} (1,0,0,1,0)$

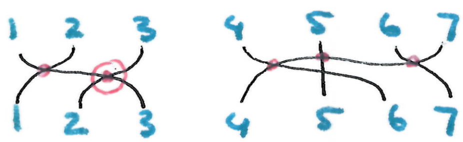
To get $I^{-1}(10010)$, build σ from 5 to 1 using (partial words): $5 \rightsquigarrow 54 \rightsquigarrow 354 \rightsquigarrow 2354 \rightsquigarrow \boxed{21354}$.

§1. Visualizing inversions:

We show two ways, based on the example $\sigma = 3126574 \xrightarrow{I} (1,1,0,3,1,0,0) \in T_7$

(I) "Wire diagram": Join i with $\sigma(i)$ in
$1 \cdots\cdot i \cdots n$
$1 \cdots \sigma(i) \cdots n$
with a wire (avoiding triple intersections)

Example:



$\mathrm{inv}(\sigma) = 6 = $ # crossings

$\boxed{\text{Crossing} \longleftrightarrow \text{inversions}}$ (Eg 3 before 2 in 1-line notation yields the marked crossing)

$\boxed{\mathrm{inv}(\sigma) = \#\text{ "wire crossing" (pairwise) in the diagram}}$

Proposition 1: $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$

Proof The wiring diagram for σ^{-1} is the one for σ, but read from bottom to top. So, the number of crossings is the same. □

(II) "Dot diagram":

(1) Write $\sigma \in \mathfrak{S}_n$ as a matrix $(P^{(\sigma)}_{ij})_{i,j} = \begin{cases} 1 & j = \sigma(i), \\ 0 & \text{else}. \end{cases}$

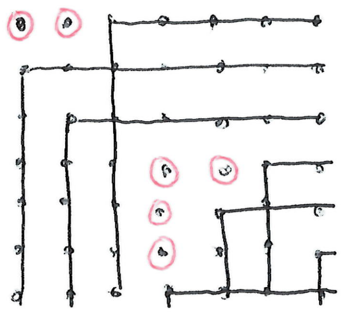
(2) Fix $n \times n$ grid of dots

(3) If $P_{ij}^{(\sigma)} = 1$, draw lines from (i,j) to the left & down.

(4) $\boxed{\text{Diagram } D_\sigma}$:= remaining dots in the grid (uncrossed ones, which we highlight by circling them)

Example: $\sigma = 3\,1\,2\,6\,5\,7\,4 \rightsquigarrow P^\sigma =$

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\rightsquigarrow D_\sigma =$ 

- Each circled dot corresponds to one of the 6 inversions in σ.
- Circled dots per column $= (1,1,3,1,0,0) = I(\sigma)$.

Proposition 2: For all $\sigma \in \mathfrak{S}_n$ & $I(\sigma) = (a_1, \ldots, a_n)$, then $a_j = \#$ circled dots in j^{th} column of D_σ for all $j = 1, \ldots, n$.

Proof: Fix $\sigma(i) = j$.

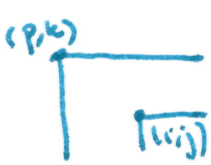
Claims: (1) j appears as the first element of an inversion $\Longleftrightarrow \exists$ circled dot in D_σ to the left of (i,j) [(j,k) inversion: $j = \sigma(i)$, $k = \sigma(p)$; $i < p$, $j > k$]

(2) j appears as the second ——— $\Longleftrightarrow \exists$ ——— on top of (i,j) [(k,j) inversion $j = \sigma(i)$, $k = \sigma(p)$; $i > p$, $j < k$]

Claim (2) gives a_j (only circled dots in col j can appear above (i,j)). $\square$

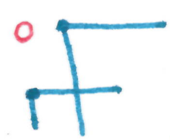
Proof of claims:

(2) is clear [$p < i$]



$p < i$ & $k < j$
(k,j) NOT inversion

vs



$p < i$ & $k > j$
(k,j) inversion.

(1) Use $D_{\sigma^{-1}}$ $\quad (j,k) \longleftrightarrow (p,i) = (\sigma^{-1}(k), \sigma^{-1}(j))$

inv for σ: $j > k$ $\qquad$ inv for σ^{-1}: $p > i$

Top in $D_{\sigma^{-1}} \longleftrightarrow$ Left in D_σ

2^{nd} pos in $D_{\sigma^{-1}} \longleftrightarrow 1^{st}$ pos in D_σ

Example:

	1st pos	2nd pos
$w_1=3$	$(3,1)$ $(3,2)$	$\emptyset$
$w_2=1$	$\emptyset$	$(3,1)$
$w_3=2$	$\emptyset$	$(3,2)$
$w_4=6$	$(6,4)$ $(6,5)$	$\emptyset$
$w_5=5$	$(5,4)$	$(6,5)$
$w_6=7$	$(7,4)$	$\emptyset$
$w_7=4$	$\emptyset$	$(7,4)$ $(5,4)$ $(6,4)$

[2 dots to the left, none on top of $(1,3)$]

⋮

§2 Inversions in multiset permutations:

Write $M=\{1^{a_1},2^{a_2},\dots,m^{a_m}\}$ multiset of $[m]$ with $a_1+\dots+a_m$ elements.

Def: An inversion of a permutation $w\in\mathfrak{S}_M$ is a 4-tuple (i,j,w_i,w_j) such that $i<j$ & $w_i>w_j$. Write $\operatorname{inv}(w):=\#$ inversions of w.

Example: (TOTAL: $\frac{4!}{2!\,2!}$)

$\mathfrak{S}_{\{1^2,2^2\}}$	inv
1122	0
1212	1
1221	2
2112	2
2121	3
2211	4

$$\sum_{w\in\mathfrak{S}_{\{1^2,2^2\}}} q^{\operatorname{inv}(w)} = 1+q+2q^2+q^3+q^4 = \begin{bmatrix}4\\2,2\end{bmatrix}_q$$

multiplicities of elements in $\{1^2,2^2\}$

Thm For $M=\{1^{a_1},\dots,m^{a_m}\}$ with $\sum_{i=1}^m a_i =: n$, $\displaystyle\sum_{w\in\mathfrak{S}_M} q^{\operatorname{inv}(w)} = \begin{bmatrix}n\\a_1,a_2,\dots,a_m\end{bmatrix}_q$

Remark: When $M=[n]$, we recover MacMahon's Thm since $[n]_q! = \begin{bmatrix}n\\1,\dots,1\end{bmatrix}_q$.

Proof. Define a map $\phi:\mathfrak{S}_M\times\mathfrak{S}_{a_1}\times\dots\times\mathfrak{S}_{a_m}\longrightarrow\mathfrak{S}_n$

$$(w_0,\ w_1,\ \dots,\ w_m)\longmapsto w$$

by • convert the a_i i's in w_0 to the numbers (consecutive): $N_i+1,\dots,\underbrace{N_i+a_i}_{=N_{i+1}}$ for $N_i=\sum_{j=1}^{i-1}a_j$ in the order specified by w_i.

• Write the transformation as an n-~~length~~-word using w_0 & above conversions.

Example : $(\overset{1\,1\,2\,2\,2\,3\,3\,3}{21331223}, 21, 231, 312) \longmapsto 42861537.$

$N_1=0,\ N_2=2,\ N_3=5$ $\qquad \mathfrak{S}_{\{1^2,2^3,3^3\}}$

$11 \xrightarrow[(21)]{} 21$

$222 \xrightarrow[(231)]{} 453$

$333 \xrightarrow[(312)]{} 867$

Obs: ϕ is a bijection (start from getting w_n from last a_n numbers in w. get location of a_n m's in w_0 from there, then proceed with w_{n-1}, etc.)

Claim: $\mathrm{inv}(w) = \mathrm{inv}(w_1) + \cdots + \mathrm{inv}(w_m) + \mathrm{inv}(w_0)$

$\uparrow$ adds to inv coming from 1^{a_1} missed by w_0.

MacMahon's Thm gives
$$[n]_q! = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} = \Big(\sum_{w_0\in\mathfrak{S}_M} q^{\mathrm{inv}(w_0)}\Big)\prod_{i=1}^{m}\underbrace{\sum_{w_i\in\mathfrak{S}_{a_i}} q^{\mathrm{inv}(w_i)}}_{=\,[a_i]_q!}$$

So
$$\sum_{w_0\in\mathfrak{S}_M} q^{\mathrm{inv}(w_0)} = \frac{[n]_q!}{[a_1]_q!\cdots[a_m]_q!} = \begin{bmatrix} n \\ a_1,\dots,a_m\end{bmatrix}_q. \qquad \square$$