Lecture XV: Generating functions, algebra of formal series

§1 Statistics via examples

Recall: Two standard ways to produce generating functions

(1) Start w/ $a_0, a_1, a_2, \ldots$ & form $\to$ ogf $= \sum_n a_n z^n$

$\to$ egf $= \sum_n \frac{a_n}{n!} z^n$

(2) Start w/ a set & sum over the elements using a statistic, eg $\sum_{\sigma \in \mathfrak{S}_n} q^{inv(\sigma)}$.

Def: A statistic on a set X is a function $\phi: X \to \mathbb{Z}$.

Examples: (1) [Rows in Pascal's Δ] $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots \binom{n}{n}, 0, 0, \ldots \Rightarrow \sum_{k=0}^{\infty} \binom{n}{k} x^k \underset{\substack{\uparrow \\ \text{Binomial Thm}}}{=} (1+x)^n$.

(2) For $A \in 2^{[n]}$, use $|A|$ as a statistic $\rightsquigarrow \sum_{A \in 2^{[n]}} x^{|A|} = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$

(3) [~~Column~~ in Pascal's Δ] Fix $k \geq 0$

$(a)_n = \binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \binom{k+3}{k}, \ldots$

Note: $\binom{k+n}{k} = \binom{n+(k+1)-1}{(k+1)-1} = \#$ of weak $(k+1)$-compositions of n

For a weak $(k+1)$-composition $\sum_{i=1}^{k+1} c_i = n$ of n, use the sum as the statistic

Then: $\sum_{k \geq 0} \binom{k+n}{k} x^n = \sum_{\substack{\text{weak} \\ (k+1)\text{-comp}}} x^n$

Thm: $\sum_{n \geq 0} \binom{k+n}{k} x^n = \frac{1}{(1-x)^{k+1}}$

Proof: $\sum_{n \geq 0} \binom{k+n}{k} x^n = \sum_{n \geq 0} \sum_{\substack{c_1 + \cdots + c_{k+1} = n \\ c_i \in \mathbb{Z}_{\geq 0}}} x^n = \sum_{n \geq 0} \sum_{\substack{|\underline{c}| = n \\ c_i \in \mathbb{Z}_{\geq 0}}} x^{c_1} x^{c_2} \cdots x^{c_{k+1}}$

$\underset{\text{in } \mathbb{C}[[x]]}{=} \underbrace{(1+x+x^2+\cdots)}_{c_1 \exp} \underbrace{(1+x+x^2+\cdots)}_{c_2 \exp} \cdots\cdots \underbrace{(1+x+x^2+\cdots)}_{c_{k+1} \exp} = \left(\frac{1}{1-x}\right)^{k+1}$ $\square$

Example (4) Can use ogf to reprove there are 2^{n-1} compositions of n with parts in $\mathbb{Z}_{\geq 1}$.

Set $a_n := \#$ compositions of n.

Then $\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \sum_{\substack{c_1 + \cdots + c_k = n \\ \text{for some } \underline{k}}} x^n = \sum_{n \geq 0} \left(\sum_{k \geq 0} \sum_{\substack{c_1 + \cdots + c_k = n \\ k \text{ fixed} \\ c_i \geq 1}} x^{c_1} \cdots x^{c_k} \right)$

$$= \sum_{n\geq 0}\sum_{k\geq 0} x^k \sum_{\substack{c_1+\dots+c_k=n\\ c_i\geq 1}} x^{(c_1-1)}\cdots x^{(c_k-1)} = \sum_{n\geq 0}\sum_{k\geq 0} x^k \sum_{\substack{c_1+\dots+c_k=n-k\\ c_i\geq 0}} x^{c_1}\cdots x^{c_k}$$

$$= \sum_{k\geq 0} x^k \sum_{n\geq 0}\sum_{\substack{c_1+\dots+c_k=n-k\\ c_i\geq 0}} x^{c_1}\cdots x^{c_k} = \sum_{k\geq 0} x^k \underbrace{\sum_{m\geq 0}\sum_{\substack{c_1+\dots+c_k=m\\ c_i\geq 0}} x^{c_1}\cdots x^{c_k}}_{(1+x+x^2+\cdots)^k} = \sum_{k\geq 0} x^k \left(\frac{1}{1-x}\right)^k$$

if $n<k$ no soln. $m=n-k\geq 0$

in $\mathbb{C}[[X]]$

$$= \sum_{k\geq 0}\left(\frac{x}{1-x}\right)^k \underset{\text{in } \mathbb{C}[[X]]}{=} \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x} = (1-x)\left(1+2x+(2x)^2+(2x)^3+\cdots\right)$$

$$= 1+(2-1)x+(2^2-2)x^2+(\underset{2\cdot 2^2}{2^3-2^2})x^3+\cdots = 1+x+2x^2+2^2x^3+2^3x^4+\cdots$$

Both series agree over $\mathbb{C}[[X]]$ iff coefficients match, so $\boxed{a_n = 2^{n-1}}$. □

Example: $(1+x+\cdots)^{-1} = 1-x$ ⇝ justifies $1+x+\cdots = \frac{1}{1-x}$.

§2 Algebra of power series:

Next, we formalize the algebraic manipulations from our examples & confirm no issues arise from

Prop (1) $\mathbb{C}[[X]]$ is a commutative ring, with no zero divisors (same if we replace $\mathbb{C}$ by a comm domain R)

(2) $f = \sum_{k\geq 0} a_k x^k$ is invertible $\iff a_0 \neq 0$ (a_0 unit in R)

Proof (1) • $f = \sum a_k x^k$, $g = \sum b_k x^k \Rightarrow f+g := \sum (a_k+b_k) x^k$.

Clear: commutative & associative. (same as for $\mathbb{C}[x]$)

• $f\cdot g = \sum_{k,\ell} a_k b_\ell x^{k+\ell} = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$ [convolution product]

Clear: commutative, distributive, associative. (extend properties of $\mathbb{C}[x]$)

• Domain: Write $f = x^n \sum_{k\geq 0} a_k x^k$ $\quad a_0\neq 0$

$g = x^m \sum_{\ell\geq 0} b_\ell x^\ell$ $\quad b_0 \neq 0$

$\Rightarrow fg = x^{n+m} \sum_{s=0}^{\infty} \underbrace{\sum_{k=0}^{s} a_k b_{s-k}}_{c_k} x^k$ has $c_0 = a_0 b_0 \neq 0$ (in $\mathbb{C}$ or R)

Def $\deg : \mathbb{C}[[X]]\setminus\{0\} \longrightarrow \mathbb{Z}_{\geq 0}$ $\quad \boxed{\deg(fg) = \deg f + \deg g}$

$f \longmapsto$ lowest order in X (n in the expression above)

(extend to 0 by $\deg(0) = \infty$)

(2) We build g so that $fg=1$. To this end we write conditions for a product of series to equal 1.

$$\left(\sum_{k\geq 0} a_k x^k\right)\left(\sum_{\ell\geq 0} b_\ell x^\ell\right) = 1$$

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n = 1 \iff c_n = \sum_{k=0}^{n} a_k b_{n-k} = \delta_{n,0} \ \forall n$$

So

$$\left.\begin{array}{l} \boxed{c_0 = a_0 b_0 = 1} \\ c_1 = a_0 b_1 + a_1 b_0 = 0 \\ c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ \vdots \\ \boxed{c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = 0 \quad \forall n\geq 1} \end{array}\right\} (*)$$

($\Rightarrow$) $c_0 = 1 = a_0 b_0$ if $fg=1$. So a_0 is a unit in $\mathbb{C}$ (or R), ie $a_0 \neq 0$ (a_0 unit in R).

($\Leftarrow$) We know the $(*)$ conditions must hold for $f^{-1} = \sum_{\ell\geq 0} b_\ell x^\ell$.
We build the coefficients b_i inductively, starting with b_0, using $(*)$

- To have $a_0 b_0 = 1$ we must choose $b_0 = a_0^{-1}$ (we can because of our hypothesis)
- Next: $a_0 b_1 + a_1 b_0 = 0$ & we know 3 of the 4 terms. We solve for b_1 to get $b_1 = -a_0^{-1} a_1 b_0$.
- Next $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \Rightarrow$ solve for b_2 : $b_2 = -a_0^{-1}(a_1 b_1 + a_2 b_0)$

In general: $a_0 b_n + \sum_{i=1}^{n} a_i b_{n-i} = 0$ & we know all terms except b_n by the inductive process. Solving for b_n gives:

$$b_n = -a_0^{-1}\sum_{i=1}^{n} a_i b_{n-i} .$$

By construction: $g = \sum_{k\geq 0} b_k x^k$ satisfies $fg = 1$ so f is invertible.

$\square$