Lecture XVI: More on formal power series & their structure

Recall: $f, g \in \mathbb{C}[\![x]\!]$ (or $\underset{R \text{ domain}}{R[\![x]\!]}$), then:

- $f+g = \sum a_k x^k + \sum b_k x^k = \sum (a_k+b_k)\, x^k$ (Term-by-Term)
- $f \cdot g = \sum\limits_{n\geq 0} \left(\sum\limits_{k=0}^{n} a_k b_{n-k}\right) x^k$ [convolution product]

- $\mathbb{C}[x]$ commutative domain (no-zero divisors)
- f is a unit in $\mathbb{C}[\![x]\!]$ $(R[\![x]\!])$ $\iff$ $a_0 \neq 0$ (unit in R)

Example: $(1+x+\cdots)^{-1}$ exists $(a_0 = 1 \neq 0)$ & $(1+x+\cdots)^{-1} = 1-x$

Method to prove $\nexists$ non-zero divisors: define $\boxed{\deg : \mathbb{C}[\![x]\!] \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}}$ with $\boxed{\text{low degree}}$

(1) $\deg(fg) = \deg f + \deg g$, (2) $\deg(f) = \infty \iff f = 0$.

§1. Convergence of sequences:

- Use degree function to define convergence of series ("in completion with (x)-adic topology")

Prop 1: A sequence $(F_n(x))_{n\in\mathbb{N}} \subseteq \mathbb{C}[\![x]\!]$ converges $\iff \lim\limits_{n\to\infty} \deg(F_{n+1}(x) - F_n(x)) = \infty$.

["Cauchy condition"]

- $F(x) = \lim\limits_{n\to\infty} F_n(x) \iff \lim\limits_{n\to\infty} \deg(F(x) - F_n(x)) = \infty$.

Prop 2: Fix sequence $(F_j(x))_j \in \mathbb{C}[\![x]\!]$, write partial sum sequence $\left(\sum\limits_{j=0}^{n} F_j(x)\right)_n$

This sequence converges to $\sum\limits_{j\geq 0} F_j(x) \iff \lim\limits_{j\to\infty} \deg F_j(x) = \infty$

Proof: $\deg F_j(x) \xrightarrow[j\to\infty]{} \infty$ means $\sum\limits_{j\geq 0} \underbrace{\sum\limits_{k=0}^{\infty} a_k^{(j)} x^k}_{=: F_j(x)} = \sum\limits_{k=0}^{\infty} \Big(\underbrace{\sum\limits_{j=0}^{\infty} a_k^{(j)}}\Big) x^k$ $\square$

[Fix k: $\exists j_0$ s.t. $a_k^{(j)} = 0$ for all $j \geq j_0$]

has finitely many non-zero terms once k is fixed. $\Rightarrow$ can sum these terms in $\mathbb{C}$.

§2. Infinite products & Compositions:

Fix $F_j(x)$ series & assume $F_j(0) := a_0^{(j)} = 1 \;\forall j$ (to avoid technical issues)

$\rightsquigarrow$ Write $\boxed{F_j := (1+G_j) \quad \deg G_j \geq 1}$

Def: $\prod\limits_{j=0}^{\infty} F_j(x) = \lim\limits_{N\to\infty} \prod\limits_{j=0}^{N} F_j(x) = \lim\limits_{N\to\infty} \underbrace{\prod\limits_{j=0}^{N} (1+G_j)}_{\in \mathbb{C}[\![x]\!]}$.

Prop 2 $\prod_{j=0}^{\infty}(1+G_j)$ with $G_j(0)=0$ converges if and only if $\deg G_j(x) \xrightarrow[j\to\infty]{} \infty$.

Pf: Again, the coefficients a_k for x^k in the products $\prod_{j=0}^{N}(1+G_j)$ stabilize as N grows.
(we are reduced to finite sums from distributive law in $\mathbb{C}[\![x]\!]$)

Def $F(x)=\sum_{n\geq 0} a_n x^n$, $G(x) \in \mathbb{C}[x]$ with $G(0)=0$, we define the composition $F(G(x)) = \sum_{n\geq 0} a_n (G(x))^n$ (If $F(x)$ is a polynomial, we can compose even if $G(0)\neq 0$)

Since $\deg G(x)^n = n \deg G(x) \geq n$, by Prop 1, the (RHS) is well defined & gives a formal series

Obs: Sum & products are compatible with compositions ($+$ & $\cdot$ for F!)

Note:
- $\sum_{n\geq 0}\frac{(1+x)^n}{n!}$ does not give a formal series $\left(F(x)=\sum_{n\geq 0}\frac{x^n}{n!} \ (=e^x),\ G(x)=1+x \quad G(0)=1\neq 0 \right)$ (infinite)
- $e^{e^x-1} = \sum_{n\geq 0}\frac{(e^x-1)^n}{n!}$ & $e^x-1=\sum_{k\geq 1}\frac{x^k}{k!} = G(x) \Rightarrow$ makes sense to write e^{e^x-1} as a formal power series

§3 **Example**: $\boxed{\text{The Binomial Series}}$

Def For any $F(x)\in\mathbb{C}[\![x]\!]$ st $F(0)=0$, pick any $\lambda\in\mathbb{C}$ & define

$$(1+F(x))^{\lambda} := \sum_{n\geq 0}\binom{\lambda}{n}(F(x))^n = \sum_{n=0}^{\infty}\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}(F(x))^n \quad \left[\text{motivated by } \lambda\in\mathbb{Z}_{\geq 0}\right]$$

Proof: This is a series because $\deg \binom{\lambda}{n}(F(x))^n \geq n\cdot\deg F(x) \geq n \Rightarrow$ use Prop 1.

Examples: (1) $(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k$ is Binomial Thm.

(2) $$\boxed{\frac{1}{(1-x)^{k+1}}} = (1+(-x))^{-k-1} = \sum_{n\geq 0}\frac{(-k-1)(-k-2)\cdots(-k-n)}{n!}(-x)^n$$

absorbe (-1) on each of the n factors

$$= \sum_{n\geq 0}\frac{(n+k)(n+k-1)\cdots(n+k-n+1)}{n!}x^n = \sum_{n\geq 0}\frac{(n+k)!}{n!\,k!}x^n = \boxed{\sum_{n\geq 0}\binom{k+n}{k}x^n}$$

Can check: $\left(\sum_{n\geq 0}\binom{k+n}{k}x^n\right)(1-x)^{k+1} = 1$.

Obs:
- Usual properties of exponentiation hold: $(1+F(x))^{\lambda+\mu} = (1+F(x))^{\lambda}(1+F(x))^{\mu}$ in $\mathbb{C}[\lambda,\mu][\![x]\!]$.
- Coeff of x^n in $(1+F(x))^{\mu}$ is a polynomial in $\mathbb{C}[\lambda]$