# Lecture XVII : Binomial Series, Exp, Logs, Formal Derivatives

**Recall**: • Use $\deg$ to define convergence of sequences in $\mathbb{C}[\![x]\!]$.
• Compositions $F(G(x))$ are elements in $\mathbb{C}[\![x]\!]$ if ① $F$ polynomial
② $F$ series & $G(0)=0$.
• **Binomial Series**: $$(1+F(x))^{\lambda} := \sum_{n=0}^{\infty} \underbrace{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}}_{\text{"}\binom{\lambda}{n}\text{"} \in \mathbb{C}[\lambda]} (F(x))^n \quad \text{for } \lambda\in\mathbb{C},\ F(0)=0$$

**Note**: This extends usual formula for $\lambda\in\mathbb{Z}_{\geq 0}$

**Corollary**: $$\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0}\binom{k+n}{k}x^n$$

Pf/ By induction on $k\in\mathbb{Z}_{\geq 0}$

• Base case: $k=0$ $\quad \frac{1}{1-x} = \sum_{n\geq 0} x^n$ ✓

• Inductive step:
$$\frac{1}{(1-x)^{k+2}} = \frac{1}{(1-x)}\,\frac{1}{(1-x)^{k+1}} = \Big(\sum_{s\geq 0} x^s\Big)\Big(\sum_{n\geq 0}\binom{k+n}{k}x^n\Big)$$

Need to check: $$\sum_{n\geq 0}\binom{k+1+n}{k+1}x^n = \sum_{n\geq 0}\Big(\sum_{m=0}^{n}\binom{k+m}{k}\Big)x^n$$

$\iff \binom{k+1+n}{k+1} = \sum_{m=0}^{n}\binom{k+m}{k}$ $\forall n$. We induct on $n$. $n=0$: $1=1$ ✓

• $n>0$: We use Pascal's binomial identity:
$$\sum_{m=0}^{n}\binom{k+m}{k} = \underset{(m=0)}{1} + \sum_{m=1}^{n}\left(\binom{k+m-1}{k}+\binom{k+m-1}{k-1}\right).$$
$$= 1+\sum_{m=1}^{n}\binom{k+m-1}{k} + \sum_{m=1}^{n}\binom{k+m-1}{k-1} = \underbrace{\sum_{s=0}^{n-1}\binom{k+s}{k}}_{=\binom{k+1+n-1}{k+1}\ (n-1<n)} + \underbrace{\sum_{m=0}^{n}\binom{(k-1)+m}{(k-1)}}_{\overset{\text{IH for } k}{=}\binom{k+n}{k}}$$
$$= \binom{k+n}{k+1} + \binom{k+n}{k} \underset{\text{Pascal's id}}{=} \binom{k+n+1}{k+1} \ ✓$$
$\boxtimes$

**Prop**: ① Usual exponentiation property holds: $(1+z)^{\mu+\lambda} = (1+z)^{\mu}(1+z)^{\lambda}$ $(\mu,\lambda\in\mathbb{C})$

Pf/ Convolution becomes Vandermonde identity: $\binom{\mu+\lambda}{n} = \sum_{k=0}^{n}\binom{\mu}{k}\binom{\lambda}{n-k}$

[Polynomial identity, valid for all $\mu,\lambda\in\mathbb{Z}_{\geq 0}$ $\Rightarrow$ valid for all $\mu,\lambda\in\mathbb{C}$]

② Extend to $(1+F(x))^{\mu+\lambda} = (1+F(x))^{\mu}(1+F(x))^{\lambda}$ $\forall F\in\mathbb{C}[\![x]\!]$ $F(0)=0$.

**Useful series**: (1) $\sum_{n\geq 0}\frac{x^n}{n!} = e^x$, $\quad$ (2) $\sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n} = \log(1+x)$

# §1 Formal Derivatives & Integration

**Def** If $F(x) = \sum_{n\geq 0} a_n x^n$, the formal derivative $F'(x)$ (or $\frac{dF}{dx}$, $DF(x)$) is the formal power series:

$$F'(x) = \sum_{n\geq 0} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n$$

**Prop**: All usual laws of differentiation apply:

(1) $(F+G)' = F' + G'$

(2) $(FG)' = F'G + FG'$

(3) $(F(G(x)))' = G'(x)\, F'(G(x))$ with $G(0)=0$.

(4) $(F^{-1})' = \frac{-F'}{F^2}$ whenever $F$ is invertible.

- Inverse operation of derivation is indefinite integration:

$$\int \sum_{n\geq 0} a_n x^n \, dx = \sum_{n\geq 0} a_n \frac{z^{n+1}}{n+1} + \text{constant}.$$

- More examples: $\sum_{n\geq 0} n a_n x^n = x F'(x)$, $\sum_{n\geq 1} n a_{n-1} x^{n-1} = (xF(x))'$
- Can use derivation & integration to shift a sequence (use egf):

$$\frac{d}{dx}\left(a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots\right) = a_1 + a_2 \frac{x^1}{1!} + a_3 \frac{x^2}{2!} + \cdots$$

- To multiply $a_i$ by $i$ apply $x\frac{d}{dx}$ to the egf:

**Example**: • Apply $x\frac{d}{dx}$ to $\frac{x}{1-x} = x + x^2 + x^3 + \cdots$ gives $x + 2x^2 + 3x^3 + \cdots$

- Apply $\left(x\frac{d}{dx}\right)^m$ to $\frac{x}{1-x}$ to get $x + 2^m x^2 + 3^m x^3 + \cdots = \sum_{j\geq 1} j^m x^j$.

- Can use operations to find a closed formula for a series.

**Example**: Let $F(x) = \sum_{n\geq 0} \binom{2n}{n} x^n$, so $a_n = \binom{2n}{n} = \frac{2n(2n-1)}{n^2} a_{n-1}$

. This gives $n a_n = 4n a_{n-1} - 2a_{n-1}$

. Compare coefficients for $x^{n-1}$ $\rightsquigarrow$ $F'(x) = 4(xF(x))' - 2F(x)$

. $\boxed{F'(x)} = 4F(x) + 4xF'(x) - 2F(x) = \boxed{4xF'(x) + 2F(x)}$

**Exercise**: $F(0)=1 \Rightarrow$ we can write $\log F(x)$ [ $\log(1+x) = \sum_{n\geq 1} (-1)^{n-1} x^n/n$ ]

So $\dfrac{F'(x)}{F(x)} = 4x + 2 \Rightarrow (1-4x)\dfrac{F'(x)}{F(x)} = 2$

Thus: $(\log F(x))' = \dfrac{F'(x)}{F(x)} = \dfrac{2}{1-4x} = -\dfrac{1}{2}(\log(1-4x))'$

This implies (by integration): $\log F(x) = -\frac{1}{2}\log(1-4x) + \text{constant}$.

$F(0) = \binom{2\cdot 0}{0} = 1$ & $(1-4x)_{(0)} = 1$ so constant $= 0$.

$$\Rightarrow \boxed{F(x) = (1-4x)^{-1/2}}$$

How to check this?

- <u>Claim 1</u>: $\binom{-1/2}{n} = \left(\frac{-1}{4}\right)^n \binom{2n}{n}$

Pf/ Follows from $r^k\left(r-\frac{1}{2}\right)^k = \dfrac{(2r)^{2k}}{2^{2k}}$ (for $r \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$)

- Use Vandermonde's Identity to conclude:

$$(-1)^n = \binom{-1}{n} = \sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = \sum_{k=0}^{n} \left(\frac{-1}{4}\right)^{k+(n-k)} \binom{2k}{k}\binom{2(n-k)}{n-k}$$
$$= \left(\frac{-1}{4}\right)^n \sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k}$$

So $\displaystyle\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$

. This is equivalent to $(F(z))^2 = \dfrac{1}{1-4z}$. ( multiply by $x^n$ & add over all $n \geq 0$)

## §2 Infinite Products

<u>Recall</u>: $\prod_{j=0}^{\infty}(1+F_j(x)) \in \mathbb{C}[[x]]$ if and only if $\deg F_j \xrightarrow[j\to\infty]{} \infty$

<u>Prop 1</u>: If $\prod_{j=0}^{\infty} A_j(x)$ & $\prod_{j=0}^{\infty} B_j(x)$ exist, then $\prod_{j=0}^{\infty} A_j B_j$ exists & $= \prod_{j=0}^{\infty} A_j \prod_{j=0}^{\infty} B_j$

<u>Proof</u>: $\deg(A_jB_j - 1) \to \infty$ if $\deg(A_j - 1) \to \infty$ & $\deg(B_j - 1) \to \infty$. So $\prod_{j=0}^{\infty} A_j B_j$ exists.

To write coefficients for a fixed $x^n$, we only involved finitely many $A_j$ & $B_j$'s so associativity for finite products gives the result. ∎

**Prop 2**: If $\prod_{i=0}^{\infty} F_i(x)$ exists, with $F_i(0)=1\ \forall i$, so does $\prod_{i=0}^{\infty} F_i^{-1}(x)$ and $\prod_{i=0}^{\infty} F_i(x) \prod_{i=0}^{\infty} F_i^{-1}(x) = 1$

**Proof**: Follows from $\deg(F_i - 1) = \deg(F_i^{-1} - 1)$ & Prop 1.

$F_i = 1 + x^k \sum_{n=0}^{\infty} a_n x^n \quad a_0 \neq 0 \qquad F_i^{-1} = 1 + x^s \sum_{n=0}^{\infty} b_n x^n \quad b_0 \neq 0$

$F_i F_i^{-1}(0) = 1$ ✓

$$\left.\begin{array}{lll} \text{if } s<k: & \text{Coeff}(F_i F_i^{-1}, x^s) = 1 \neq 0 \\ k<s & \text{———}(\text{———}, x^k) = 1 \neq 0 \end{array}\right\} \Rightarrow \boxed{s=k}$$ □

**Application**:
$$\boxed{\prod_{i\geq 1}(1+x^i) = \frac{1}{\prod_{i\geq 1}(1-x^{2i-1})}}$$
(useful for gen functions of partitions)

**Proof**: $\prod_{i\geq 1}(1-x^i)$ exists $\underset{\text{Prop 2}}{\Longrightarrow} \frac{1}{\prod_{i\geq 1} 1-x^i}$ exists

$\frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-x^i)}$ also exists. We view it in 2 ways:

(1) Combine Prop 1 & 2 to write it as $\prod_{i\geq 1} \frac{(1-x^{2i})}{(1-x^i)} = \prod_{i\geq 1}(1+x^i)$

(2) Write $\prod_{i\geq 1}(1-x^{2i})$ as $1\cdot(1-x^2)\cdot 1\cdot(1-x^4)\cdot 1\cdot(1-x^6)\cdots\cdots$

So $\frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-x^i)} = \frac{1}{\prod_{i\geq 1}(1-x^{2i-1})}$ (odd exponents in denom are the only survivors).

We use (1) = (2) to conclude the identity holds. ☒