# Lecture XVIII: Fibonacci Numbers & Fibonacci Polynomials

**Question**: How many compositions of $n$ ($x_1+\cdots+x_k=n$, $x_i \geq 1\ \forall i$ for some $k$) are there using only 1's and 2's as parts?

**Examples**

| $n$ | compositions (full list) | TOTAL |
|---|---|---|
| $n=1$ | $1$ | 1 |
| $n=2$ | $1+1$, $2$ | 2 |
| $n=3$ | $1+1+1$, $1+2$, $2+1$ | 3 |
| $n=4$ | $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$ | 5 |
| $n=5$ | $1+1+1+1+1$, $1+1+1+2$, $1+1+\underbrace{2}+1$, $1+2+1+1$, $2+1+1+1$, $1+2+\underbrace{2}$, $2+1+\underbrace{2}$, $2+2+1$ | 8 (3+5) |

**Recurrence?** Every composition ends with 1 or 2.

- If ending w/ 1, get comp of $n-1$ by removing last part
- If " w/ 2, ——— $n-2$ ———

**Prop 0**: Let # of compositions of $n$ using 1's & 2's $=\tilde{F}_n$. Then, $\tilde{F}_n$ satisfies
$$\tilde{F}_n=\tilde{F}_{n-1}+\tilde{F}_{n-2} \quad \& \quad \tilde{F}_1=1,\ \tilde{F}_2=2.$$

**Def**: The Fibonacci numbers are defined as:
$$F_0=1,\quad F_1=1,\quad \& \quad F_n=F_{n-1}+F_{n-2} \quad \forall n \geq 2$$

(**Note**: Usually we set $F_0=0$, $F_1=1$, so things will be shifted ~~forward~~)

$\tilde{F}_3=1+2$ vs $F_2=2$, $F_3=3$. $\Rightarrow$ $\boxed{\tilde{F}_n=F_n}$.

**Q**: What's the ogf for $F_n$?

$$\sum_{n\geq 0} F_n x^n=\underset{\substack{n\geq 0\\ \text{(Total value}\\ \text{of comp.)}}}{\sum}\ \sum_{k\geq 0}\ \sum_{\substack{a_1+\cdots+a_k=n\\ a_i\in\{1,2\}}} x^n \underset{\boxed{\text{swap order of sum}}}{=} \sum_{k\geq 0}\ \sum_{\substack{a_1+\cdots+a_k\\ a_i\in\{1,2\}}} x^{a_1+\cdots+a_k}$$

[for $n=0$, only contribution comes from $k=0$]

( $\sum_n F_n x^n$ is a valid element in $\mathbb{C}[[x]]$ & $\sum_a x^{|a|}$ ) has deg $\geq k \to \infty$

$$=\sum_{k\geq 0}\underset{\substack{\downarrow\\ a_1 \exp}}{(x+x^2)}\underset{\substack{\downarrow\\ a_2 \exp}}{(x+x^2)}\cdots\underset{\substack{\downarrow\\ a_k \exp}}{(x+x^2)}=\sum_{k\geq 0}(x+x^2)^k=\frac{1}{1-(x+x^2)}$$

$G(x)=x+x^2$ deg $\geq 1$.

**Obs**: This ogf encodes the Fibonacci recurrence
$$\sum F_n x^n=\frac{1}{1-x-x^2} \iff (1-x-x^2)\Big(\sum_{n\geq 0} F_n x^n\Big)=1$$

$\Leftrightarrow \quad F_0 + (F_1 - F_0)x + (F_2 - F_1 - F_0)x^2 + (F_3 - F_2 - F_1)x^3 + \dots$

$\qquad + (F_n - F_{n-1} - F_{n-2})x^n + \dots \qquad = 1$

meaning $\quad F_0 = 1$

$\qquad F_1 - F_0 = 0 \quad \rightsquigarrow F_1 = 1$

$\qquad F_n - F_{n-1} - F_{n-2} = 0 \quad \forall n \geq 2 \quad \rightsquigarrow F_n = F_{n-1} + F_{n-2}.$

Recap: $(F_n)_n$ satisfies $\quad F_0 = 1, F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2.$

$$\sum_{n \geq 0} F_n x^n = \frac{1}{1 - x - x^2}$$

Key observation was to find a disjoint union:

$$\text{Comp}_n = \left\{ \begin{array}{l} \text{comp. of } n \\ \text{using 1's \& 2's} \end{array} \right\} \longleftrightarrow \underbrace{\left\{ \begin{array}{l} \text{comp of } n-1 \\ \text{using 1's \& 2's} \end{array} \right\}}_{\text{(add 1 on the right)}} \sqcup \underbrace{\left\{ \begin{array}{l} \text{comp of } n-2 \\ \text{using 1's \& 2's} \end{array} \right\}}_{\text{(add 2 on the right)}}$$

Let's introduce a statistic related to this:

$\phi : \text{Comp}_n \longrightarrow \mathbb{Z} \qquad \phi(a_1 + \dots + a_k) = |\{i : a_i = 1\}|$ = number of 1's in comp.

Def: The $n^{th}$ Fibonacci polynomial is $F_n(x) := \sum_{c \in \text{Comp}_n} x^{\phi(c)}$.

Two questions: Q1: Is $F_n(x)$ related to the Fibonacci sequence? A: YES.

Q2: What are the coefficients of $F_n(x)$?

Note: $F_n(1) = \sum_{c \in \text{Comp}_n} 1 = F_n$. (by Coreman's Thm) $\rightsquigarrow$ Answer to Q1.

Prop 1: $F_n(x) = x F_{n-1}(x) + F_{n-2}(x)$

Proof: $F_n(x) = \sum_{a \in \text{Comp}_n} x^{\#\{i : a_i = 1\}} = \sum_{b \in \text{Comp}_{n-1}} x^{\#\{i : b_i = 1\} + 1} + \sum_{c \in \text{Comp}_{n-2}} x^{\#\{i | c_i = 1\}}$

last element of $\underline{a}$ $\qquad$ [last elem of $\underline{a} = 2$]

$= x F_{n-1}(x) + F_{n-2}(x)$ ■

Prop 2: $F_n(x) = \sum_{j=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n-j}{j} x^{n-2j}$. $\rightsquigarrow$ diagonal coeffs in Pascal's Triangle

| | | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| n=1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| n=2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| n=3 | 2 | 1 | 2 | 1 | 0 | 0 | 0 |
| n=4 | 3 | 1 | 3 | 3 | 1 | 0 | 0 |
| n=5 | 4 | 1 | 4 | 6 | 4 | 1 | 0 |
| n=6 | 5 | 1 | 5 | 10 | 10 | 5 | 1 |

Proof: Use Prop 1 & induction on $n$.

Base cases: $n = 0 \quad F_0(x) = 1 \quad \text{Comp}_0 = \emptyset \quad \lceil \frac{0-1}{2} \rceil = \lceil -\frac{1}{2} \rceil = 0$

$\qquad n = 1 \quad F_1(x) = x^1 \quad \text{Comp}_1 = \{1\} \quad \lceil \frac{1-1}{2} \rceil = \lceil 0 \rceil = 0$

[ Check: $n = 2 \quad F_2(x) = x^2 + 1 \quad \text{Comp}_2 = \{1+1, 2\} \quad \lceil \frac{2-1}{2} \rceil = \lceil \frac{1}{2} \rceil = 1. \quad \& \ F_2 = x F_1 + F_0$ ✓ ]

Inductive Step: We assume the statement is true for $F_k$ with $k<n$. Thus:

$$xF_{n-1} = x\sum_{j=0}^{\lceil\frac{n-1-1}{2}\rceil}\binom{n-1-j}{j}x^{n-1-2j} = \sum_{j=0}^{\lceil\frac{n-2}{2}\rceil}\binom{n-1-j}{j}x^{n-2j}$$

$$F_{n-2} = \sum_{j=0}^{\lceil\frac{n-3}{2}\rceil}\binom{n-2-j}{j}x^{n-2-2j} \underset{k=j+1}{=} \sum_{k=1}^{\lceil\frac{n-3}{2}\rceil+1}\binom{n-1-k}{k-1}x^{n-2k}$$

We analyze 2 cases, depending on the parity of $n$. In both cases, use Prop 1.

CASE 1: $n=2l$

$\lceil\frac{n-3}{2}\rceil+1 = \lceil\frac{2(l-1)-1}{2}\rceil+1 = l-1+1 = l$

$\lceil\frac{n-2}{2}\rceil = \lceil\frac{2(l-1)}{2}\rceil = l-1$ & $\lceil\frac{n-1}{2}\rceil = \lceil\frac{2l-1}{2}\rceil = l$

$$F_n = xF_{n-1}+F_{n-2} = \sum_{j=0}^{l-1}\binom{n-1-j}{j}x^{n-2j} + \sum_{j=1}^{l}\binom{n-1-j}{j-1}x^{n-2j}$$

$$= \underset{(j=0)}{x^n} + \sum_{j=1}^{l-1}\underbrace{\left(\binom{n-1-j}{j}+\binom{n-1-j}{j-1}\right)}_{=\binom{n-j}{j}\text{ by Pascal's recursion}} x^{n-2j} + \underbrace{\binom{n-1-l}{l-1}}_{=\binom{2l-1-l}{l-1}=\binom{l-1}{l-1}=1} x^{n-2l}$$

& $\binom{n-l}{l}=\binom{l}{l}=1$

$$= \sum_{j=0}^{l}\binom{n-j}{j}x^{n-2j} \quad \& \quad l=\left\lceil\frac{n-1}{2}\right\rceil$$

CASE 2: $n=2l+1$

$\lceil\frac{n-3}{2}\rceil+1 = \lceil\frac{2(l-1)-1}{2}\rceil+1 = l-1+1 = l$

$\lceil\frac{n-2}{2}\rceil = \lceil\frac{2(l-1)+1}{2}\rceil = l = \lceil\frac{2l}{2}\rceil = \lceil\frac{n-1}{2}\rceil$

As before: $$F_n = xF_{n-1}+F_{n-2} = \sum_{j=0}^{l}\binom{n-1-j}{j}x^{n-2j} + \sum_{j=1}^{l}\binom{n-1-j}{j-1}x^{n-2j}$$

$$= \underset{(j=0)}{x^n} + \sum_{j=1}^{l}\binom{n-j}{j}x^{n-2j} = \sum_{j=0}^{l}\binom{n-j}{j}x^{n-2j} \quad \& \quad l=\left\lceil\frac{n-1}{2}\right\rceil$$

∎

Note: For a direct combinatorial proof, enough to show:

$$\#\{\text{comp of } n \text{ using 1's \& 2's with } (n-2j) \text{ 1's}\} = \binom{n-j}{j} \quad \forall\, j=0,\dots,\left\lceil\frac{n-1}{2}\right\rceil$$

Pf/ $a \in$ (LHS) If ends in 1 $\Rightarrow a = a'1$, $a' \in$ (LHS) for $n-1$ & $(n-1-2j)$ 1's

If ends in 2 $\Rightarrow a = a'2$, $a' \in$ (LHS) for $n-2$ & $(n-2j)$ 1's

$n-2j = (n-2)-2(j-1)$

Argue by double induction on $(n,j)$

• $n=0$ or $j=0$ (LHS) is $\emptyset$ or $\{\underbrace{1+\cdots+1}_{n \text{ times}}\}$ so we get $\binom{n-j}{j} = \begin{cases} 0 & \text{for } j>0 \\ 1 & \text{for } j=0. \end{cases}$ & $\binom{n-0}{0}=1$.

• The recursion + inductive hypothesis gives

$$\#(\text{LHS}) = \underbrace{\binom{n-1-2j}{j-1}}_{\text{ending in } 1} + \underbrace{\binom{n-2-2(j-1)}{j-1}}_{\text{ending in } 2} = \binom{n-1-2j}{j-1} + \binom{n-2j}{j-1} \underset{\substack{\text{Pascal's} \\ \text{recurrence}}}{=} \binom{n-2j}{j}$$

$\square$

**Obs**: if $j > \lceil \frac{n-1}{2} \rceil$ we have $\binom{n-j}{j} = 0$ $\left[ \begin{matrix} n=2l \\ n=2l+1 \end{matrix} \rightsquigarrow j \geq l+1 \text{ \& so } n-j<j \right.$.
$\Rightarrow$ We can omit the upper limit in the sum.

**Last question**: Q3: Can we extend $\sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2}$ to $F_n(t)$? ($t$ = any variable but $x$)

A: YES!

**Prop 3**: $1 + tx + (t^2+1)x^2 + (t^3+2t)x^3 + (t^4+3t^2+1)x^4 + \cdots = \sum_{n\geq 0} F_n(t) x^n = \frac{1}{1-tx-x^2}$

**Proof 1**: $(1-tx-x^2) \sum_{n\geq 0} F_n(t) x^n = F_0(t) + (F_1(t) - tF_0(t))x +$
$+ (-F_0(t) - tF_1(t) + F_2(t))x^2 + \sum_{n\geq 3} (-F_{n-2}(t) - tF_{n-1}(t) + F_n(t)) x^n$

$F_0 = 1$

$F_1(t) - tF_0(t) = t - t = 0$

$F_2(t) - tF_1(t) - F_0(t) = t^2 + 1 - t(t) - 1 = 0$

$F_n(t) - tF_{n-1}(t) - F_{n-2}(t) = 0 \quad \forall n \geq 2$ by Prop 1.

So product is 1, as we wanted. ☒

**Proof 2**: $\sum_{n\geq 0} F_n(t) x^n = \sum_{k\geq 0} \underbrace{(tx+x^2)^k}_{\substack{\text{comps w/ } k \text{ parts} \\ \text{using 1's \& 2's} \\ \text{write a "t" for each 1.}}} \stackrel{\downarrow}{=} \frac{1}{1-(tx+x^2)} = \frac{1}{1-tx-x^2}$ ☒

composition of gen series & $G(x) = tx+x^2$ & $G(0)=0$

$\left[ \begin{matrix} a_i = 1 \rightsquigarrow t \cdot x \\ a_i = 2 \rightsquigarrow x^2 \end{matrix} \right]$

Use same "distributive Law" & $\binom{n-j}{j} = \#\{a_1+\cdots+a_k \in \text{Comp}_n$ with $(n-2j)$ 1's$\}$