Lecture XIX: Recurrence relations & ordinary generating functions (o.g.f.)

Recall: 1. $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $\forall n$ (Recursion)

$F_n = \#\{a_1 + \dots + a_k = n \quad a_i = 1 \text{ or } 2 \ \forall i\}$ Fibonacci Numbers.

2. Showed $\displaystyle\sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2}$.

We can use partial fractions to give a closed formula for F_n (Binet's Formula from Lecture 1)

3. Factor $1-x-x^2 = (1-\phi x)(1-\bar{\Phi} x)$ $\quad \phi = \frac{1+\sqrt{5}}{2}$, $\bar{\Phi} = 1-\phi = \frac{1-\sqrt{5}}{2}$

4. $\displaystyle\sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2} = \frac{\alpha}{1-\phi x} + \frac{\beta}{1-\bar{\Phi}x} \underset{\substack{\uparrow \\ \text{geom series}}}{=} \sum_{n\geq 0} (\alpha\phi^n + \beta\bar{\Phi}^n)\, x^n$

Conclude $F_n = \alpha\phi^n + \beta\bar{\Phi}^n$ $\forall n$

Use $F_0 = 1$ & $F_1 = 1$ to solve for α & β $\rightsquigarrow$ $\alpha = \frac{\phi}{\sqrt{5}}$, $\beta = \frac{-\bar{\Phi}}{\sqrt{5}}$

Binet's Formula: $$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} \quad \text{for all } n.$$

Note: denominator encodes reflected recurrence $[F_n - F_{n-1} - F_{n-2} \rightsquigarrow 1-x-x^2]$

Claim: We can use the same method to find close formulas for any sequence admitting a linear recurrence relation over $\mathbb{C}$ (or any alg. closed field of char 0, eg. $\overline{\mathbb{C}[t]}$)

Def: Let $c(z) = 1 + c_1 z + c_2 z^2 + \dots + c_d z^d \in \mathbb{C}[z]$ of degree $= d \geq 1$ $(c_d \neq 0)$

We define the reflected polynomial $c^R(z) = z^d + c_1 z^{d-1} + c_2 z^{d-2} + \dots + c_d$, $c_d \neq 0$

Obs: $c(z) = z^d c^R(\frac{1}{z})$

In particular, if $c^R(z) = (z-\alpha_1)\cdots(z-\alpha_d)$ (we allow $\alpha_i = \alpha_j$)

$$\Rightarrow c(z) = z^d\left(\tfrac{1}{z} - \alpha_1\right)\cdots\left(\tfrac{1}{z}-\alpha_d\right) = (1-\alpha_1 z)\cdots(1-\alpha_d z)$$

In the example above $c(z) = 1 - z - z^2 = (1-\phi z)(1-\bar{\Phi} z)$ & $c^R(z) = z^2 - z - 1 = (z-\phi)(z-\bar{\Phi})$

The roots of c^R become the powers in the relevant summands for F_n. This will be true in general.

The following is the main theorem in this subject:

Theorem: Fix a sequence $c_1, \ldots, c_d$ of d complex numbers with $c_d \neq 0$ & set $C(z) = 1 + c_1 z + \cdots + c_d z^d = \prod_{i=1}^{k} (1-\alpha_i z)^{d_i}$ where $\alpha_1, \ldots, \alpha_k$ are the roots of $c^R(z)$ (with $\mathrm{mult}(\alpha_i, c^R) = d_i$).

Consider a counting function $F : \mathbb{N}_0 \longrightarrow \mathbb{C}$ & write $F_n := F(n)$ $\forall n$.

The following conditions are equivalent:

(A1) [Recurrence of order d] $F_{n+d} + c_1 F_{n+d-1} + \cdots + c_d F_n = 0$ $\forall n \geq 0$

(A2) [Generating function] $F(z) = \sum_{n\geq 0} F_n z^n = \dfrac{P(z)}{C(z)}$

where $P \in \mathbb{C}[z]$ has $\deg(P) < d$.

(A3) [Partial fractions] $F(z) = \sum_{n\geq 0} F_n z^n = \sum_{i=1}^{k} \dfrac{g_i(z)}{(1-\alpha_i z)^{d_i}}$

where $g_1, \ldots, g_k \in \mathbb{C}[z]$ has $\deg(g_i) < d_i$ $\forall i = 1, \ldots, k$.

(A4) [Explicit form] $F_n = \sum_{i=1}^{k} P_i(n) \alpha_i^n$

where $P_1(n), \ldots, P_k(n) \in \mathbb{C}[n]$ $\deg P_i(n) < d_i$ $\forall i = 1, \ldots, k$.

Proof: We define 4 sets, one for each (Ai) condition:

$$V_i := \{F : \mathbb{N}_0 \longrightarrow \mathbb{C} : (F_n)_n \text{ satisfies (Ai)}\} \quad i = 1, \ldots, 4.$$

- Each V_i is a $\mathbb{C}$-vector space (functions are a v.s & conditions (Ai) are linear in $(F_n)_n$.)
- $\dim_{\mathbb{C}} V_i = d$. $\forall i = 1, \ldots, 4$

Pf/ For V_1: $f(0), \ldots, f(d-1)$ can be chosen arbitrarily & the rest are determined

Basis $\{\underbrace{(1,0,\cdots 0)}_{d}, \underbrace{(0,1,0\cdots 0)}_{d}, \cdots \underbrace{(0,\cdots 0,1)}_{d}\} \Rightarrow V_1 \simeq \mathbb{C}^d$

For V_2: f is determined by $P(z) = c_0 + c_1 z + \cdots + c_{d-1} z^{d-1}$, $c_0, \ldots c_{d-1} \in \mathbb{C}$

Again: $V_2 \simeq \mathbb{C}^d$ $P \longleftrightarrow (c_0, \ldots, c_{d-1})$.

For V_3: Use $(\underset{d_1}{\mathrm{coeff}\, g_1}, \ldots, \underset{d_k}{\mathrm{coeff}\, g_k})$ & $\sum_{i=1}^{k} d_i = d$ by construction.

So $V_3 \simeq \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k} \simeq \mathbb{C}^d$.

For V_4: Use $(\underset{d_1}{\mathrm{coeff}\, P_1}, \ldots, \underset{d_k}{\mathrm{coeff}\, P_k}) \Rightarrow V_4 \simeq \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k} \simeq \mathbb{C}^d$

Strategy: To show (Ai)'s are all equivalent, we must show $V_1 = V_2 = V_3 = V_4$.

By dimension, enough to show $V_2 \subseteq V_1$, $V_3 \subseteq V_2$, $V_3 \subseteq V_4$.

<u>$V_2 \subseteq V_1$</u>: Pick $f \in V_2$. Then $c(z)F(z) = P(z)$

$$\sum_{n\geq 0}^{\infty}(1+c_1z+\dots+c_dz^d)F_n z^n = \sum_{n=0}^{\infty}\Big(\sum_{i=0}^{d} F_{n+d-i}\,c_i\Big)z^{n+d} + \sum_{n=0}^{d-1} A_n z^n \qquad (c_0=1) \quad \Big[A_n=\sum_{k=0}^{n} F_k c_{n-k}$$

Write $P(z) = a_0 + a_1 z + \dots + a_{d-1}z^{d-1} + 0\cdot z^d + 0\,z^{d+1} + \dots$

This gives $F_{n+d} + F_{n+d-1}c_1 + \dots + F_n c_d = 0 \;\; \underline{\forall n}$ (coeff z^{n+d})

So $f \in V_1$.

<u>$V_3 \subseteq V_2$</u>: Pick $f \in V_3$. Then:

$$\sum_{n=0}^{\infty} F_n z^n = \sum_{i=1}^{k}\frac{g_i(z)}{(1-\alpha_i z)^{d_i}} = \frac{\overbrace{\sum_{i=1}^{k} \underbrace{g_i(z)}_{\deg<d_i} \underbrace{\prod_{j\neq i}(1-\alpha_j z)^{d_j}}_{\deg=\sum_{j\neq i} d_j}}^{\deg<d}}{\underbrace{\prod_{i=1}^{k}(1-\alpha_i z)^{d_i}}_{=c(z)}} = \frac{P(z)}{c(z)}$$

and $\deg P \leq \max_i \Big(\underbrace{\deg g_i}_{<d_i} + \sum_{j\neq i} d_j\Big) < \max_i \Big(\sum_{j=1}^{k} d_j\Big) = d$

So $f \in V_2$.

<u>$V_3 \subseteq V_4$</u>: Pick $f \in V_3$ & look at $\dfrac{g_i(z)}{(1-\alpha_i z)^{d_i}}$ for each $i=1,\dots,k$.

The Binomial Series (Lecture 17) gives

$$\frac{1}{(1-\alpha_i z)^{1+(d_i-1)}} = \sum_{n\geq 0}\binom{d_i+n-1}{n}\alpha_i^n z^n = \sum_{n\geq 0}\binom{d_i+n-1}{d_i-1}\alpha_i^n z^n$$

Multiply by $g_i(z) = g_0^{(i)} + g_1^{(i)} z + \dots + g_{d_i-1}^{(i)} z^{d_i-1}$ gives:

$$\frac{g_i(z)}{(1-\alpha_i z)^{d_i}} = \sum_{n\geq 0}\Big(\sum_{j=0}^{d_i-1} g_j^{(i)}\binom{d_i+n-j-1}{d_i-1}\alpha_i^{n-j}\Big)z^n$$

$$= \sum_{n\geq 0}\underbrace{\Big(\sum_{j=0}^{d_i-1}\frac{g_j^{(i)}}{\alpha_i^j}\binom{d_i+n-j-1}{d_i-1}\Big)}_{=:P_i(n)}\alpha_i^n z^n$$

<u>Note</u>: $\binom{d_i+n-j-1}{d_i-1} \in \mathbb{C}[n]$ has degree $= d_i-1 \implies \deg P_i(n) \leq d_i - 1$.

Conclude $\displaystyle\sum_{n=0}^{\infty} F_n z^n = \sum_{n=0}^{\infty}\Big(\sum_{i=1}^{k} P_i(n)\alpha_i^n\Big)z^n$ & so $F_n = \displaystyle\sum_{i=1}^{k} P_i(n)\alpha_i^n \;\; \forall n$

So $f \in V_4$ $\blacksquare$