Lecture XX: Recurrence relations & their combinatorial interpretations

· **Thm**: 4 equivalences for counting functions $f : \mathbb{N}_0 \to \mathbb{C}$ (or $\overline{\mathbb{C}(t)}, \overline{\mathbb{Q}}, \ldots$) $F_n := f(n)$

given $C(x) = 1 + c_1 x + \cdots + c_d x^d \in \mathbb{C}[x]$ $\quad c_d \neq 0$ with $C_R(x) = x^d C(\frac{1}{x}) = \prod_{i=1}^{k} (x - \alpha_i)^{d_i}$

(A1): $F_{n+d} + c_1 F_{n+d-1} + \cdots + c_d F_n = 0 \quad \forall n \geq 0,$

(A2) $\quad F(x) = \sum_{n \geq 0} F_n x^n = \dfrac{P(x)}{C(x)} \qquad \deg P(x) < d \qquad P \in \mathbb{C}[x],$

(A3) $\quad F(x) = \sum_{i=1}^{k} \dfrac{g_i(x)}{(1 - \alpha_i x)^{d_i}} \qquad \deg g_i(x) \leq d_i \qquad g_i \in \mathbb{C}[x],$

(A4) $\quad F_n = \sum_{i=1}^{k} p_i(n)\, \alpha_i^n \qquad \deg p_i(n) < d_i \qquad p_i \in \mathbb{C}[n].$

Example 1: Say $(a_n)_{n \in \mathbb{N}}$ is defined by $a_0 = 0$, $a_1 = 1$, $a_n = 6a_{n-1} - 9a_{n-2}$. (A1)

$\rightsquigarrow C(x) = 1 - 6x + 9x^2 \rightsquigarrow C_R(x) = x^2 - 6x + 9 = (x-3)^2$ $\quad [k=1,\ d_1 = 2 = d,\ \alpha_1 = 3]$

(A4) gives $a_n = (\alpha + \beta n)\, 3^n \quad \forall n \Rightarrow$ need to solve for α & β.

$$\begin{cases} 0 = a_0 = \alpha \\ 1 = a_1 = (\alpha + \beta)\, 3 \end{cases} \rightsquigarrow \alpha = 0, \quad \beta = \frac{1}{3}$$

Conclude: $\boxed{a_n = n\, 3^{n-1} \quad \forall n}$

(A2) $F(x) = \sum_{n \geq 0} a_n x^n = \dfrac{A + Bx}{1 - 6x + 9x^2} \rightsquigarrow$ solve for A & B.

· $a_0 = F(0) = A$ & $a_1 = F'(0) = \left.\dfrac{B(1 - 6x + 9x^2) - (A + Bx)(-6 + 18x)}{(C(x))^2}\right|_{x=0} = \dfrac{B + 6A}{1}$

So $A = a_0 = 0$, $B = a_1 - 6A = a_1 = 1$.

$$\Rightarrow \boxed{F(x) = \frac{x}{1 - 6x + 9x^2}}$$

Alternative: $C(x) F(x) = A + Bx$, where $C(x) F(x) = a_0 + (a_1 - 6a_0)x + \cdots$ $\Big\} \Rightarrow$ $A = a_0$, $B = a_1 - 6a_0$ as above.

Remark: $\mathbb{C}$ in theorem can be replaced by any alg. closed field of char $= 0$

Ex (L.18): $(F_n(t))_n$ Fib polynomials: $F_0 = 1$, $F_1 = t$ $\quad \boxed{F_n - tF_{n-1} - F_{n-2} = 0}$ & $\sum_{n=0}^{\infty} F_n(t) x^n = \dfrac{1}{1 - tx - x^2}$

Example 2: [Other fields]

Fix $q \in \mathbb{Z}_{\geq 2}$ & define a q-sequence $(a_n)_n$ by $\begin{cases} a_0 = 1, \ a_1 = q \\ \boxed{a_n - q\, a_{n-1} + a_{n-2} = 0} \end{cases}$ (A1)

Ex: $q=2 \rightsquigarrow 1, 2, 3, 4, 5, \ldots$ (all $\mathbb{Z}_{>0}$)

$q=3 \rightsquigarrow 1, 3, 8, 21, 55, \ldots$ (every other Fibonacci #'s)

- Think of q as a formal parameter $\Rightarrow a_n \in \mathbb{C}[q]$. $\forall n$
- $C(X) = 1 - qX + X^2 \rightsquigarrow C^R(x) = x^2 - qx + 1 = (x - \gamma_1(q))(x - \overline{\gamma}_1(q))$

$\gamma_{(q)} := \dfrac{q + \sqrt{q^2-4}}{2}$, $\overline{\gamma}_{(q)} := \dfrac{q - \sqrt{q^2-4}}{2}$ $\in \overline{\mathbb{C}[q]}$

(A2) $F = \sum_{n \geq 0} a_n X^n = \dfrac{A + BX}{1 - qX + X^2}$ $\rightsquigarrow$ solve for A & B use $F(0)$ & $F'(0)$

$F(0) = a_0 = A$

$F'(0) = a_1 = B - A(-q) = B + Aq$

$\Bigg\} \rightsquigarrow \begin{array}{l} 1 = A \\ q = B + q \Rightarrow B = 0 \end{array}$

$\Rightarrow F(x) = \sum_{n \geq 0} a_n X^n = \dfrac{1}{1 - qX + X^2}$

- Can check this for $q=2$: $\sum_{n \geq 0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) x^n = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n \right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ ✓

(A4) $a_n = P_1(n)\, \gamma^n + P_2(n)\, \overline{\gamma}^n$ $\quad$ $\deg P_1 < 1$ & $\deg P_2 < 1 \Rightarrow$ constants P_1, P_2 (if $q \neq 2 \rightsquigarrow \gamma \neq \overline{\gamma}$)

Solve for P_1 & P_2 with a_0 & a_1

$1 = a_0 = P_1 + P_2$

$q = a_1 = P_1 \left(\dfrac{q + \sqrt{q^2-4}}{2}\right) + P_2 \left(\dfrac{q - \sqrt{q^2-4}}{2}\right)$

$\Bigg\} \Rightarrow P_1 = \dfrac{\gamma}{\sqrt{q^2-4}}$ & $P_2 = \dfrac{-\overline{\gamma}}{\sqrt{q^2-4}}$

So $\boxed{a_n = \dfrac{1}{\sqrt{q^2-4}} \left(\gamma_{(q)}^{n+1} - \overline{\gamma}_{(q)}^{n+1} \right) \quad \forall n \geq 0, \ q \neq 2}$

If $q=2$: $1 = \gamma = \overline{\gamma}$ $d_1 = 2$.

$a_n = (a + bn)$ $\forall n \geq 0$

$1 = a_0 = a$

$2 = a_1 = (a+b) \rightsquigarrow b = 2 - 1 = 1$

$\boxed{a_n = 1 + n \quad \forall n \geq 0}$

§1 Combinatorial Interpretations:

$C(x) = 1 + c_1 x + \cdots + c_d x^d$ with $c_d \neq 0 \rightsquigarrow F_{n+d} + c_1 F_{n+d-1} + \cdots + c_d F_n = 0$

Claim: We can always relate this to compositions using $1, \ldots, d$ only.

Ex 1: $C(x) = 1 - X - X^2 - \cdots - X^d \rightsquigarrow F_n := \#\{ a_1 + \cdots + a_k = n \quad a_i \in \{1, \ldots, d\}\}$

$$\sum_{n\geq 0} F_n x^n = \sum_{n\geq 0}\sum_{k=0}^{\infty}\sum_{\substack{a_1+\cdots+a_k=n\\ a_i\in\{1,\dots,d\}}} x^n = \sum_{k=0}^{\infty}\sum_{\substack{a_1,\dots,a_k\\ a_i\in\{1,\dots,d\}}} x^{a_1+\cdots+a_k} = \sum_{k=0}^{\infty}\Big(\sum_{a_1} x^{a_1}\Big)\cdots\Big(\sum_{a_k} x^{a_k}\Big)$$

$$= \sum_{k=0}^{\infty}(x+x^2+\cdots+x^d)^k \underset{\substack{\uparrow\\ \text{geom series}}}{=} \frac{1}{1-x-x^2-\cdots-x^d} \quad \text{as we wanted!}$$

Q: What if we have $c(x) = 1-2x-3x^2$?
Think of compositions with 2 types of 1's & 3 types of 2's ("colored 1's & 2's")

$$\Rightarrow \sum_{n\geq 0} F_n x^n = \sum_{k\geq 0}(x+x+x^2+x^2+x^2)^k = \sum_{k\geq 0}(2x+3x^2)^k \underset{\substack{\uparrow\\ \text{geom series}}}{=} \frac{1}{1-2x-3x^2}$$

Prop 1: The ogf for compositions of n using $\tilde{c}_1$ "colors" of 1, $\tilde{c}_2$ "colors" of 2, ...
$\tilde{c}_d$ "colors" of d is: $\displaystyle\sum_{k\geq 0}(\tilde{c}_1 x^1+\tilde{c}_2 x^2+\cdots+\tilde{c}_d x^d)^k = \frac{1}{1-\tilde{c}_1 x-\cdots-\tilde{c}_d x^d}$
$(\tilde{c}_1,\dots,\tilde{c}_d\in\mathbb{Z}_{\geq 0})$

Q2: What if we have positive coefficients in $c(x)$?

Ex: $$\frac{1}{1-2x+x^2} = \frac{1}{1-(2x-x^2)} \underset{\substack{\uparrow\\ \text{geom.}}}{=} \sum_{k\geq 0}(2x^1-x^2)^k \qquad \text{How to interpret this?}$$

A: Think of compositions with 2 colors of 1's & 1 color of 2's but weighted

Ex: weight $= (-1)^{\#2\text{'s in comp}}$.

Prop 2: If $b_n =$ weighted #{compositions of n using $1,\dots,d$ as parts} $\forall n$
where $wt(i) = (-\beta_i)^{\#i\text{'s}}$

$$= \sum_{\substack{a_1+\cdots+a_k\\ \text{comp of } n\\ \text{using } 1,\dots,d}} (-\beta_1)^{\#1\text{'s}}\cdots(-\beta_d)^{\#d\text{'s}}.$$

Then ogf $$\sum_{n\geq 0} b_n x^n = \frac{1}{1+\beta_1 x+\cdots+\beta_d x^d} = \sum_{k\geq 0}(-\beta_1 x-\cdots-\beta_d x^d)^k$$

From Thm we conclude that b_n satisfies the recurrence $b_{n+d}+\beta_1 b_{n+d-1}+\cdots+\beta_d b_n=0$

Ex: $F_n(t) = \displaystyle\sum_{\substack{\text{comp of } n\\ \text{with } 1\text{'s}\,\&\,2\text{'s}}} t^{\#1\text{'s}}$ (weighted count) & $c(x) = 1-tx-x^2$; $\displaystyle\sum_n F_n(t)x^n = \sum_{k\geq 0}(tx+x^2)^k$