Lecture XXI: Non-linear recurrences & partition identities

§1: Recurrences can be non-linear. When $\underline{n}$ is involved we use exponential generating functions.

$(a_n = \text{inv}(\mathfrak{S}_n))$

Ex: (a_n) defined by $a_0 = 1$, $a_1 = 1$, $a_{n+1} = a_n + n a_{n-1}$ $\forall n \geq 1$

- Start from $a_{n+1} = a_n + n a_{n-1}$ & multiply by $\frac{x^n}{n!}$ & $\sum_{n\geq 0}$

$$\sum_{n=0}^{\infty} a_{n+1}\frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} n a_{n-1}\frac{x^n}{n!} \qquad (*)$$

Write $F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ $\Rightarrow$ $\sum_{n=0}^{\infty} a_{n+1}\frac{x^n}{n!} = F'(x)$

$$\sum_{n=0}^{\infty} n a_{n-1}\frac{x^n}{n!} = x\sum_{n=1}^{\infty} a_{n-1}\frac{x^{n-1}}{(n-1)!} = xF(x)$$

$\Rightarrow (*)$ gives $F'(x) = F(x) + xF(x) = (1+x)F(x)$

- $(\ln F(x))' = \frac{F'(x)}{F(x)} = (1+x) \Rightarrow \ln F(x) = x + \frac{x^2}{2} + \text{constant}.$

(HW4)

Evaluate at $x=0$: $F(0) = a_0 = 1$ so $0 = \ln 1 = 0 + \text{constant} \Rightarrow \boxed{\text{const}=0}$

To finish: exponentiate: $\boxed{F(x) = e^{x+\frac{x^2}{2}}}$

⚠ didn't use $a_1 = 1$ it is forced on us since $a_1 = F'(0) = (1+x)F(x)\big|_{x=0} = 1\cdot a_0$

Note: We can reverse the process! I.e. exp. gen. fun knows about the recurrence

Assume $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = e^{(x+\frac{x^2}{2})}$

Take $\frac{d}{dx}$: $\underbrace{\sum_{n=0}^{\infty} a_{n+1}\frac{x^n}{n!}}_{} = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!} = (1+x)e^{x+\frac{x^2}{2}} = (1+x)\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{n!} + \frac{a_{n-1}}{(n-1)!}\right)x^n = \sum_{n=0}(a_n + n a_{n-1})\frac{x^n}{n!}$$

Equating coefficients: $\boxed{a_{n+1} = a_n + n a_{n-1} \ \forall n \geq 1}$

x^0: $a_1 = a_0$

Value for a_0 comes from $a_0 = e^{x+\frac{x^2}{2}}\Big|_{x=0} = e^0 = \boxed{1}$.

☒

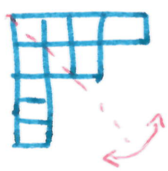
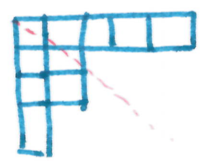
§3 Partition identities

Recall $P_{\leq k}(n) = \#\{\text{partitions } \lambda \vdash n \text{ with } \leq k \text{ parts}\}$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$, $\sum \lambda_i = n$

Prop: $$\sum_{n\geq 0} P_{\leq k}(n)\, q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{j=1}^{k} \frac{1}{1-q^j}.$$

Proof: Recall the involution on partitions (flip the young diagram of λ along it's diagonal to get λ')

Ex: $\lambda = (4,3,1,1,1)$ $\qquad$ $\lambda' = (5,2,2,1)$

$\lambda =$  $\longrightarrow$ $\lambda' =$ 

$\lambda_1 = 4$ & 5 parts $\qquad$ $\lambda'_1 = 5$. & 4 parts

If λ has $\leq k$ parts, all parts in λ' are $\leq k$.

$$\sum_{n\geq 0} P_{\leq k}(n)\, q^n = \sum_{n\geq 0} \sum_{\substack{\lambda \vdash n \\ \lambda \text{ w/} \leq k \text{ parts}}} q^{|\lambda|} \underset{\text{inv.}}{=} \sum_{n\geq 0} \sum_{\substack{\lambda' \vdash n \\ \lambda' \text{ w/ largest} \\ \text{part} \leq k}} q^{|\lambda'|} = \sum_{n\geq 0} \sum_{\substack{m_1 + 2m_2 + \cdots + k m_k = n \\ m_i \geq 0}} q^{m_1 + 2m_2 + \cdots + k m_k}$$

(mult of i in λ') $[\lambda' = 1^{m_1} 2^{m_2} \cdots k^{m_k}]$

$$= \Big(\sum_{m_1 \geq 0} q^{m_1}\Big)\Big(\sum_{m_2\geq 0} q^{2m_2}\Big)\cdots\Big(\sum_{m_k \geq 0} q^{k m_k}\Big)$$

$$= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdots \frac{1}{1-q^k}. \qquad \square$$

Corollary (Euler's Partition Thm)

$$\sum_{n\geq 0} P(n)\, q^n = \prod_{j\geq 1} \frac{1}{(1-q^j)}$$

Proof: Note: (RHS) is well def. since $\deg\big((1-q^j)-1\big) \xrightarrow[j\to\infty]{} \infty$. & $\Big(\prod_{j\geq 1}(1-q^j)\Big)^{-1} = \prod_{j\geq 1} \frac{1}{1-q^j}$

Equality follows by letting $k \to \infty$ in Prop above:

$$\lim_{k\to\infty} \sum_{n\geq 0} P_{\leq k}(n)\, q^n = \sum_{n\geq 0} P(n)\, q^n \qquad \blacksquare$$

$P_{\leq k}(n) \xrightarrow[k\to\infty]{} P(n)$ (as soon as $k > n$ $P_{\leq k}(n) = P(n)$)

Q: What else can we do? $\qquad P(n,k) = P_{\leq k}(n) - P_{\leq k-1}(n)$ [exactly k parts]

Prop 2: $$\sum_{n\geq 0} P(n,k)\, q^n = \frac{q^k}{(1-q)\cdots(1-q^k)}$$

Prop 3: Let $S=(S_1,S_2,\dots)$ with $S_i\subseteq \mathbb{Z}_{\geq 0}$ $\forall i$ & define

$\mathcal{P}(S)=\{\lambda \text{ partition where } m_i(\lambda):=\text{mult}(i,\lambda)\in S_i \ \forall i\}$

Consider the generating function in the variables $q=(q_1,q_2,\dots)$

$$F(S;q)=\sum_{\lambda\in\mathcal{P}(S)} q_1^{m_1(\lambda)}\, q_2^{m_2(\lambda)}\cdots q_i^{m_i(\lambda)}\cdots$$

Then: $$F(S,q)=\prod_{i\geq 1}\Big(\sum_{j\in S_i} q_i^{\,j}\Big)$$

Proof: . Every summand of $F(S,q)$ is a monomial involving finitely many q_i's.

. Every monomial comes from a ! partition λ. $\Rightarrow$ coefficient of a fixed $\underline{q}^{\underline{m}}$ in $F(S,q)$ is always 1

$\Rightarrow F(S,q)\in\mathbb{C}[[q_1,q_2,\dots]]$ formal series & formula holds by inspection. ■

Corollary: If $P(S,n)=\#\{\lambda\vdash n : \lambda\in\mathcal{P}(S)\}$, then $\sum_{n\geq 0}P(S,n)q^n=\prod_{i\geq 1}\sum_{j\in S_i} q^{ij}$.

Pf/ Set $q_i:=q^i$ in Prop 3. ■
$n=1m_1+2m_2+\dots$

§3 Exploiting conjugation $\lambda\to\lambda'$

Prop 4: If λ partition, then $\sum_{i\geq 1}(i-1)\lambda_i=\sum_{i\geq 1}\binom{\lambda_i'}{2}$.

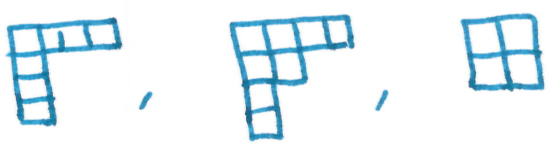
Proof: Use Young diagram & put $(i-1)$ on each box in i^{th} row (λ_i boxes).

. Row sum gives (LHS).

Ex: λ: $\begin{array}{cccc|l} \lambda_1' & \lambda_2' & \lambda_3' & \lambda_4' & \\ \hline 0 & 0 & 0 & 0 & \lambda_1 \\ 1 & 1 & 1 & & \lambda_2 \\ 2 & & & & \lambda_3 \end{array}$

. Col sum $=$ (RHS) [λ_i' boxes on col i with entries $0,1,\dots,\lambda_i'-1$ $\Rightarrow$ Total $\binom{\lambda_i'}{2}=\frac{\lambda_i'(\lambda_i'-1)}{2}$] □

Q: Are there any $\lambda=\lambda'$?

A: Many! Eg  , etc. ⇝ diagonal hooks 

Def: $\mathrm{Fix}(n):=\{\lambda\vdash n : \lambda=\lambda'\}$ (fixed pts of involution $\lambda\to\lambda'$) **Q**: What is $|\mathrm{Fix}(n)|$?

Prop: $$\sum_{n\geq 0}|\mathrm{Fix}(n)|q^n=(1+q)(1+q^3)(1+q^5)\cdots=\prod_{i\geq 1}(1+q^{2i-1})$$

Proof We use diagonal hooks of λ to create a new partition $\mu\vdash n$, $\mu_i=\#$ boxes in i^{th} hook (odd!)

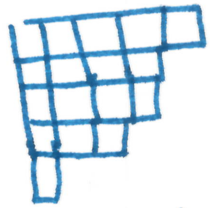
By construction: $\mu_1>\mu_2>\dots$

Construct $\phi: \mathrm{Fix}(n)\longrightarrow\{\mu\vdash n \ \ \mu_i \text{ odd } \forall i \ \ \mu_1>\mu_2>\dots>0\}$

$\lambda \longmapsto (\mu_i=\#\text{ boxes on } i^{th} \text{ hook of } \lambda)$

<u>Claim</u>: ϕ is a bijection $\quad (\lambda = \lambda' \Rightarrow \mu_i$ is odd $\forall i$, $\mu_i \neq \mu_j \ \forall i \neq j)$

<u>Example</u>: $\lambda = (5,4,4,3,1)$



$\longmapsto \mu = (9,5,3)$

Inverse map: $\operatorname{len}(\mu)$ gives # hooks & $\lambda_i = \frac{\mu_i - 1}{2} + i$ for all $i \leq \operatorname{len}(\mu)$
Symmetry of λ gives the rest of λ_j.

<u>Now</u> $\quad |\mathrm{Fix}(n)| = |\{\mu \vdash n$ only odd parts & <u>all</u> <u>distinct</u> $\}|$

$\Rightarrow$ Set $S_i = \begin{cases} \{0,1\} & \text{if } i \text{ odd} \\ \{0\} & \text{if } i \text{ even} \end{cases}$ $\quad \longrightarrow$ mult ≤ 1 for <u>odd</u> parts & $= 0$ for even parts.

Corollary for $P(S,n) = |\mathrm{Fix}(n)|$ gives

$$\sum_{n \geq 0} |\mathrm{Fix}(n)| q^n = \sum_{n \geq 0} P(S,n) q^n = \prod_{i \geq 0} \Big(\sum_{j \in S_i} q^{ij} \Big)$$

$$= \underset{(i=0)}{1} \; \underset{i=1}{(1+q)} \; \underset{i=2}{1} \; \underset{i=3}{(1+q^{3\cdot 1})} \cdots\cdots$$

$$= \prod_{i \geq 1} (1 - q^{2i-1}) \qquad \blacksquare$$