Lecture XXII: More partition identities

Recall: $P(n, \leq k) = \#\{\lambda \vdash n \text{ with } \leq k \text{ parts}\} = P_{\leq k}(n)$; $P(n) = \#\{\lambda \vdash n\}$

$P(n, \leq k, \leq j) = \#\{\lambda \vdash n$ ———— $\& \ \lambda_1 \leq j\} \rightsquigarrow$ young diagrams fitting in $k \times j$ grid with n boxes.

Props. $\displaystyle\sum_{n \geq 0} P_{\leq k}(n)\, q^n = \prod_{j=1}^{k} \frac{1}{(1-q^j)} \ \underset{k \to \infty}{\overset{\lim}{\rightsquigarrow}} \ \sum_{n \geq 0} P(n)\, q^n = \prod_{j \geq 1} \frac{1}{(1-q^j)}$ [Euler's Partition Thm]

• Refinement 1: $\boxed{\displaystyle\sum_{n \geq 0} P(n,k)\, q^n = \frac{q^k}{\prod_{j=1}^{k}(1-q^j)}}$

Refinement 2 → next page

• Further refinement: restrict multiplicities! $\lambda = \langle 1^{m_1} \cdots j^{m_j} \rangle$ $\quad m_i := \text{mult}(i, \lambda) \ \forall i \geq 0$

$S = (S_1, S_2, \ldots)$ $\quad \mathcal{P}(S) := \{\lambda \text{ part with } m_i(\lambda) \in S_i \ \forall i\}$

Prop: $\displaystyle\sum_{\lambda \in \mathcal{P}(S)} q_1^{m_1(\lambda)} \cdot q_2^{m_2(\lambda)} \cdots = \prod_{i \geq 1} \Big(\sum_{j \in S_i} q_i^{\,j}\Big)$

→ **Corollary**: If $P(S,n) = \#\{\lambda \vdash n : \lambda \in \mathcal{P}(S)\} \Rightarrow \boxed{\displaystyle\sum_{n \geq 0} P(S,n)\, q^n = \prod_{i \geq 0}\Big(\sum_{j \in S_i} q^{ij}\Big)}$

Application: $\text{Fix}(n) = \{\lambda \vdash n \quad \lambda = \lambda'\}$

$\left(S_j = \begin{cases} \{0,1\} & , j \text{ odd} \\ \{0\} & j \text{ even} \end{cases}\right.$ $\simeq \{\mu \vdash n$ w/ only odd & $\neq$ parts$\}$ $\Rightarrow \displaystyle\sum_{n \geq 0} |\text{Fix}(n)|\, q^n = \prod_{i \geq 1}(1+q^{2i-1})$

Euler's odd/distinct Theorem: Set $P_d(n) = \#\{\lambda \vdash n$ with distinct parts$\}$
$P_{odd}(n) = \#\{\lambda \vdash n$ " only odd parts$\}$

Thm: $P_d(n) = P_{odd}(n) \ \forall n$

Example: $n=7$ $\quad \{7, 61, 52, 43, 421\}$ vs $\{7, 511, 331, 31111, 1^7\}$ Both $\# = 5$

Proof 1: $\displaystyle\sum_{n \geq 0} P_d(n)\, q^n \underset{\substack{S_i = \{0,1\} \ \forall i \\ \text{in Corollary}}}{=} \prod_{i \geq 1}(1+q^i) = \prod_{i \geq 1}\left(\frac{1-q^{2i}}{1-q^i}\right) = \frac{\prod_{i \geq 1}(1-q^{2i})}{\prod_{i \geq 1}(1-q^i)}$

$\displaystyle\underset{\substack{\text{cancel all} \\ (1-q^{2m})}}{=} \frac{1}{1-q}\,\frac{1}{1-q^3} \cdots = \prod_{i \geq 1} \frac{1}{(1-q^{2i-1})} \underset{\substack{\text{geom} \\ \text{series}}}{=} \prod_{i \geq 1}\Big(\sum_{j \geq 0}(q^{2i-1})^j\Big) =$

$\displaystyle= \prod_{i \text{ odd}}\Big(\sum_{j \geq 0} q^{ij}\Big) \underset{S_i = \begin{cases} \mathbb{N}_0 & i \text{ odd} \\ \{0\} & i \text{ even}\end{cases}}{=} \sum_{n \geq 0} P_{odd}(n)\, q^n.$

Power series agree $\Rightarrow$ equate term by term to get $\boxed{P_d(n) = P_{odd}(n) \ \forall n}$

Obs: M. Bousquet-Mélou & K. Eriksson: "Lecture Hall partitions", The Ramanujan Journal 1 (1) (1997), 101–111.

Thm [BM-E] Coefficient of q^n in $F_{k}(q) = \frac{1}{(1-q)} \cdots \frac{1}{(1-q^{2k-1})}$ [finite product in Series above] is # of "Lecture Hall" partitions: non-neg integer solutions to

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_k}{k} \quad \text{with} \quad \sum_{i=1}^{k} \lambda_i = n$$

Proof 2 [Bijective] Define $\phi : \{\lambda \vdash n \text{ in } \underline{\text{odd}} \text{ parts}\} \longrightarrow \{\mu \vdash n \text{ in } \underline{\text{distinct}} \text{ parts}\}$

$\lambda = \prod_j (2j-1)^{r_j} \in$ Domain, set $\phi(\lambda) = \mu$, where $\forall j$:

$m = (2j-1)\, 2^k \in \mu \iff 2^k$ appears in binary expansion of r_j

Claim: ϕ is a bijection & target μ has distinct parts. Also $\mu \vdash n$.

(Every pos integer has a ! expression $(2j-1)\,2^k$ with $k \geq 0$, $j \geq 1$)

Example: $\lambda = \langle 1^3\, 3^2\, 5^{12}\, 9^5 \rangle \vdash 114$

$$114 = 1\,(\underbrace{1+2}_{3}) + 3(\underbrace{2^1}_{1}) + 5\,(\underbrace{2^3+2^2}_{12}) + 9\,(\underbrace{2^2+2^0}_{5})$$

$$= 1 + 2 + 6 + 40 + 20 + 36 + 9 \quad \rightsquigarrow \mu = (40, 36, 20, 9, 6, 2, 1)$$

$\square$

Refinement 2:

$$\boxed{\sum_{n \geq 0} P(n, \leq k; \leq j)\, q^n = \begin{bmatrix} j+k \\ j \end{bmatrix}_q}$$

Proof 1: Induction on $(j+k) \geq 0$ + Pascal's q-recurrence: $\begin{bmatrix} m \\ j \end{bmatrix}_q = \begin{bmatrix} m-1 \\ j \end{bmatrix}_q + q^{m-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q$

- $j+k=0 \rightsquigarrow P(n, \leq 0, \leq 0) = \delta_{n,0}$ & $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$
- **Inductive Step:** $m = j+k$ in recurrence gives:

$$\begin{bmatrix} j+k \\ j \end{bmatrix}_q = \begin{bmatrix} j+(k-1) \\ j \end{bmatrix}_q + q^k \begin{bmatrix} (j-1)+k \\ j-1 \end{bmatrix}_q$$

$$\underset{\text{IH}}{=} \sum_{n \geq 0} P(n, \leq k-1, \leq j)\, q^n + \sum_{n \geq 0} P(n, \leq k, \leq j-1)\, q^{n+k}$$

$P(n, \leq k, \leq j-1) = P(n+k, k, \leq j)$, $\mu \mapsto (\mu_1 - 1, \ldots, \mu_k - 1)$.

$P(n+k, k, \leq j) = P(n+k, \leq k, \leq j) - P(n+k, \leq k-1, \leq j)$

$$= \sum_{n \geq 0} P(n, \leq k-1, \leq j)\, q^n + \sum_{m \geq k} P(m, \leq k, \leq j)\, q^m - \sum_{m \geq k} P(m, \leq k-1, \leq j)\, q^m$$

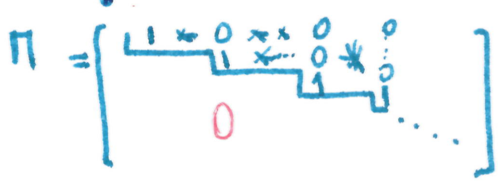
$$= \sum_{n=0}^{k-1} P(n, \leq k-1, \leq j)\, q^n + \sum_{n \geq k} P(n, \leq k, \leq j)\, q^n$$

But $P(n, \leq k-1, \leq j) = P(n, \leq k, \leq j)$ $\forall$ $n = 0, \ldots, k-1$ since $P(n, k, \leq j) = 0$ in this range.

<u>Conclude</u>: $\begin{bmatrix} j+k \\ j \end{bmatrix}_q = \sum_{n=0}^{k-1} P(n, \leq k, \leq j)\, q^n + \sum_{n \geq k} P(n, \leq k, \leq j)\, q^n = \sum_{n\geq 0} P(n, \leq k, \leq j)\, q^n$

<u>Proof 2</u>: Use $\begin{bmatrix} m \\ n \end{bmatrix}_q = \#\{ V \subseteq \mathbb{F}_q^m \ \dim V = n\}$ for $q = p^j$ for any $j \in \mathbb{Z}_{\geq 1}$. $\square$

Given $V \subseteq \mathbb{F}_q^{j+k}$ $\wedge$ $\dim V = k$ write a matrix $M = \left[\begin{array}{c} v_1 \\ \vdots \\ v_k \end{array}\right] \in \mathbb{F}_q^{k \times (j+k)}$ in <u>row-reduced echelon form</u>, so $M =$



etc.

<u>Example</u>: $j+k = 7$, $k = 4$ $\quad M = \begin{bmatrix} 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \rightsquigarrow \underline{a} = (1, 3, 4, 6)$

$[* \in \mathbb{F}_q$ arbitrary$]$ $\qquad \lambda = (3, 2, 2, 1)$

$(a_1, \ldots, a_k) :=$ columns where first non-zero entry on each row appears

$\Rightarrow \lambda_i := \#\{*\text{'s in row}_i \text{ of } M\} = j - a_i + i$ $\quad (= (j+k) - a_i - (k-i))$

<u>Obs</u> $\lambda = (\lambda_1, \ldots \lambda_k)$ defines a partition because $a_1 < a_2 < \cdots < a_k \leq j+k$

- $n := \sum_{i=1}^{k} \lambda_i = |\lambda|$
- at <u>most</u> k parts
- $\lambda_1 = j - a_1 + 1 \leq j$

<u>Q</u>: How many matrices give this λ? <u>A</u> $q^{|\lambda|}$ (q options for each $*$)

Conversely, each $\lambda \in \mathcal{P}(; \leq k, \leq j)$ determines $\underline{a}$ uniquely & gives $q^{|\lambda|}$ matrices.

<u>Conclude</u>: $\begin{bmatrix} j+k \\ k \end{bmatrix} = \sum_{\lambda \in \mathcal{P}(; \leq k ; \leq j)} q^{|\lambda|} = \sum_{n \geq 0} P(n, \leq k ; \leq j)\, q^n.$ $\square$

§2 More ids: <u>Euler's Pentagonal Thm</u>:

<u>Note</u>: $\prod_{i \geq 1} (1 - z^i) = 1 - z - z^2 + 0 \cdot z^3 + 0 \cdot z^4 + z^5 + z^7 - z^{12} - z^{15} + z^{22} + z^{26} \pm \cdots$

- Coefficients $= 0, 1, -1$?
- Nonzero coefficients of x^i $i > 0$ $= \underset{-}{1, 2}\, ; \underset{+}{5, 7}\, ; \underset{-}{12, 15}\, ; \underset{+}{22, 26}\, ; \ldots$

$\Rightarrow$ Conj pattern $= \left(\frac{3j^2 - j}{2}, \frac{3j^2 + j}{2} \right)$ with sign $(-1)^j$. $= \left[\left\{ \frac{3j^2 + j}{2} \quad j \in \mathbb{Z} \right\} \ \text{sign} = (-1)^j \right.$

Euler's Pentagonal Theorem: $\displaystyle\prod_{i\geq 1}(1-z^i) = 1+\sum_{j\geq 1}(-1)^j\left(z^{\frac{3j^2-j}{2}} + z^{\frac{3j^2+j}{2}}\right).$

$\displaystyle\left(= 1+\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} (-1)^j z^{\frac{3j^2+j}{2}} \right)$

Proof [Bressoud & Zeilberger]: Write $\displaystyle\prod_{i\geq 1}(1-z^i) = \sum_{n\geq 0} a_n z^n$

Know by Euler's Partition Thm: $\displaystyle\left(\sum_{n\geq 0} a_n z^n\right)\sum_{n\geq 0} P(n) z^n = 1.$

$\displaystyle\sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k P(n-k)\right) z^n$

So (*) $\boxed{\displaystyle\sum_{k=0}^{n} a_k P(n-k) = 1 \quad \forall n\geq 1}$ & $a_0 P(0) = 1$ ✓ by construction.

<u>To show</u>: $a_k = \begin{cases} 1 & \text{if } k = \frac{3j^2+j}{2} \text{ \& } j \text{ even} \\ -1 & \text{"} \quad k = \frac{3j^2+j}{2} \text{ \& } j \text{ odd} \\ 0 & \text{else} \end{cases}$

($\underline{\text{Pentagonal}}_{j>0}$: "$\frac{3j^2-j}{2}$" = #(dots in nested pentagons up to side length j))

Set $b_j := \frac{3j^2+j}{2}$ for $j\in\mathbb{Z}$.

Then (*) is equivalent to $\displaystyle\sum_{j \text{ even}} P(n-b(j)) - \sum_{j \text{ odd}} P(n-b(j)) = 0 \quad \forall j\in\mathbb{Z}$

To prove this, we build an explicit bijection (involution $\phi: \bigcup_j \mathrm{Par}(n-b(j))$ ↻)

$$\phi: \bigcup_{j \text{ even}} \mathrm{Par}(n-b(j)) \longrightarrow \bigcup_{j \text{ odd}} \mathrm{Par}(n-b(j))$$

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_k) \longmapsto \mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_t)$$

where $\mu = \begin{cases} k+3j-1, \lambda_1 - 1, \dots, \lambda_k - 1 & \text{if } k+3j \geq \lambda_1 \quad (\text{length} \leq k+1) \\ \lambda_2+1, \dots, \lambda_k+1, \underbrace{1, \dots, 1}_{\lambda_1 - 3j - k - 1} & \text{if } k+3j < \lambda_1 \quad \text{length} \leq \lambda_1 - 3j - 2 \end{cases}$

- If first case: $|\mu| = n - b_j + 3j - 1 = n - (b_j - 3j + 1) = n - b_{j-1}$ (j & $j-1$ ≠ parity)
- If second case: $|\mu| = n - b_j + (k-1) - 3j - k - 1 = n - (b_j + 3j + 2) = n - b_{j+1}$ (j & $j+1$ ——)

<u>Check</u>: $\phi_1 : \mathrm{Par}(n - b_p)_{k+3j\geq\lambda_1} \longrightarrow \mathrm{Par}(n - b_{p-1})_{k+3j<\mu_1}$ is an involution $\begin{pmatrix}\phi_1\circ\phi_2 = id \\ \phi_2\circ\phi_1 = id\end{pmatrix}$

$\phi_2 : \mathrm{Par}(n - b_p)_{k+3j<\lambda_1} \longrightarrow \mathrm{Par}(n - b_{p+1})_{k+3j\geq\mu_1}$

<u>Example</u>: $n=12$, $j=-2$ $\Rightarrow b_{-2} = \frac{3\cdot 4-2}{2} = 5$ $\quad \lambda = 3211 \in \mathrm{Par}(7) = \mathrm{Par}(12-5)$ $(k=4)$

$b_{-1} = \frac{3\cdot 1-1}{2} = 1$

- $k + 3\cdot j = 4-6 = -2 < \lambda_1 = 3$ $\Rightarrow \lambda_1 - 3j - k - 1 = 3+6-4-1 = 4$

$\Rightarrow \mu = \phi(\lambda) = 3221111 \in \mathrm{Par}(11) = \mathrm{Par}(12 - b(-1))$ & $k' = 7$ satisfies $4 = k' + 3(-1) \geq \mu_1 = 2$

$\Rightarrow \phi(\mu) = (7-3-1, 2, 1, 1, 0\,0\,0\,0) = (3,2,1,1) = \lambda$