Lecture XXIII: Catalan numbers, Dyck paths & Catalan recurrences

§1 Catalan numbers

- Example of a recurrence of convolution type $\rightsquigarrow$ A000108 in OEIS.

Def: $(C_n)_n$ Catalan numbers are defined by $\begin{cases} C_0 = 1 \\ C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0 = \sum_{i=0}^{n} C_i C_{n-i} & \forall n \geq 0 \end{cases}$

Examples

n	0	1	2	3	4	5	6	7	8	9
C_n	1	1	2	5	14	42	132	429	1430	4862

Q (1) Combinatorial interpretations?
(2) Closed formulas?

Obs: The recurrence is equivalent to the functorial equation

$$C(x) = 1 + x\,(C(x))^2$$

for the generating function $C(x) = \sum_{n\geq 0} C_n x^n$.

Prop: $C(x) = \dfrac{1-\sqrt{1-4x}}{2x} = \dfrac{2}{1+\sqrt{1-4x}}$

Pf/ Use quadratic formula $\rightsquigarrow \dfrac{1 \pm \sqrt{1-4x}}{2x} = \dfrac{2}{1 \mp \sqrt{1-4x}}$

$C(0) = 1$ forces only 1 soln: $+$ in denominator $\Rightarrow$ $-$ in numerator. $\square$

Thm: $C_n = \dfrac{1}{n+1}\dbinom{2n}{n} \quad \forall n \geq 0$

Alternative: $C_n = \dbinom{2n}{n} - \dbinom{2n}{n+1}$. $-\frac{1}{2}$ $\frac{-n+3}{2}$

Proof: Expand $(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4)^n x^n = \sum_{n \geq 0} \dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n x^n$

$$= 1 + \sum_{n\geq 1} \left(\frac{1}{2}\right)^n \frac{1\,\overbrace{(-1)(-3)\cdots\cdots(-2n+3)}^{n-1 \text{ many}}}{n!}(-1)^n 4^n x^n$$

$$= 1 + \sum_{n\geq 1} 2^n \frac{(2n-3)(2n-5)\cdots\cdots(3)(1)(-1)}{n!} x^n$$

$$= 1 - \sum_{n\geq 1} \frac{2}{n}\, 2^{n-1}\frac{(2n-3)(2n-5)\cdots\cdots 3\cdot 1}{(n-1)!} x^n$$

$$= 1 - \sum_{n\geq 1} \frac{2}{n} \frac{(2n-2)(2n-3)(2n-4)(2n-5)\cdots\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}{(n-1)!\,(n-1)!} x^n$$

$2^{n-1}(n-1)! = \prod_{\substack{m \text{ even} \\ 2\leq m \leq 2n-2}} m$

$$= 1 - \sum_{n\geq 1} \frac{2}{n}\frac{(2(n-1))!}{(n-1)!\,(n-1)!} x^n = 1 - \sum_{n\geq 1} \frac{2}{n}\binom{2n-2}{n-1} x^n$$

$$\Rightarrow \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 1} \frac{1}{n}\binom{2(n-1)}{n-1} x^{n-1} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

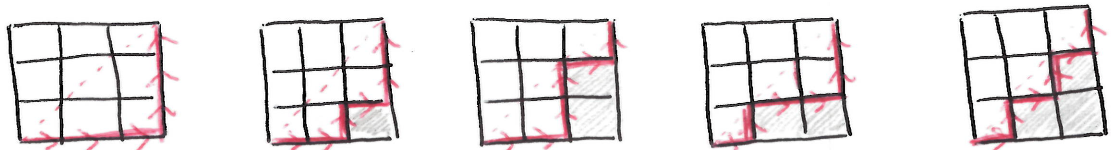
Conclude: $C_n = \frac{1}{n+1}\binom{2n}{n} \quad \forall n \geq 0$ ∎

§1 Combinatorial Interpretations:

① Dyck paths:

Def: A Dyck path of length $2n$ is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) using steps $N=(0,1)$ & $E=(1,0)$, whose edges lie below the line $x=y$

Example:



$C_3 = 5$

col height $(0,0,0)$ $(0,0,1)$ $(0,0,2)$ $(0,1,1)$ $(0,1,2)$

Thm: The number of Dyck paths of length $2n$ is C_n.

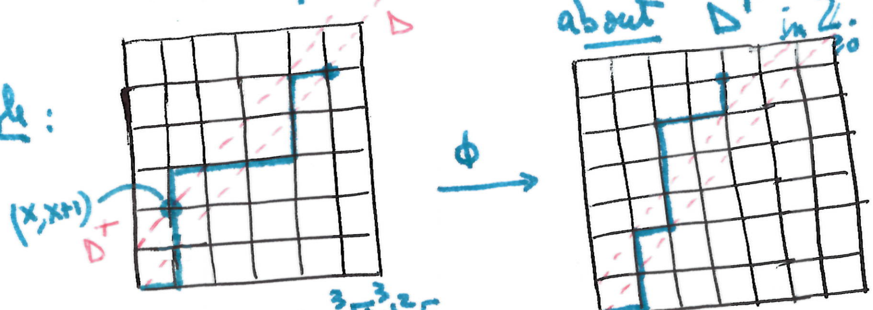
Proof: [André's Reflection Method]

- Pick all paths in grid with nN & nE steps. $\rightsquigarrow \binom{2n}{n}$ many
- Good paths = do not cross diagonal
- Bad paths = cross diagonal and touch the next higher diagonal (D^+)

If w is a bad path, reflect portion of w after this first bad point $(x,x+1)$ about D^+ in $\mathbb{Z}^2_{\geq 0}$. Call $\phi(w)$ the new path

Example:

$n=5$



$w = EN^3EN^3E^2E$

end pt $=(5,5)$

$5N$ & $5E$

$\phi(w) = EN^2EN^3E^2N$

end pt $=(4,6)$

$6N$ & $4E$

Reflection method gives $\phi(w)$ path of length $2n$ from $(0,0)$ to $(n-1,n+1)$

$(n+1)N$ $(\Rightarrow (n-1)E)$

& crosses D^+.

$(n-x)$ E in $w \Leftrightarrow (n-x)$ N in $\phi(w)$ after $(x,x+1)$

$+ (x+1)N$ before $(x,x+1)$ $\Rightarrow (n+1)N$

Note: ϕ is a bijection: $\{$Bad paths nE & nN from $(0,0)$ to $(n,n)\}$ $\longrightarrow$ $\{$paths of length $2n$ with $(n+1)N$ from $(0,0)$ to $(n-1,n+1)\}$

$\binom{2n}{n+1}$ many!

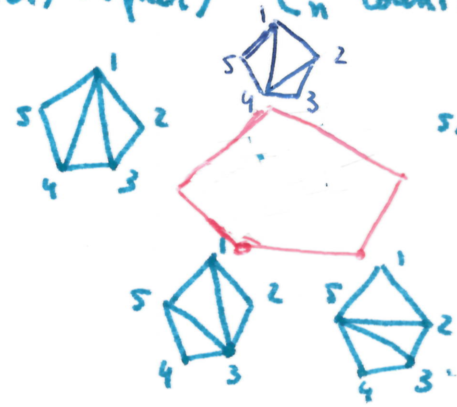
Conclude: Good paths = TOTAL − bad paths = $\binom{2n}{n} - \binom{2n}{n+1} = C_n$ $\square$

Obs: Don't cross D if and only if no initial segment of the word has more N's than E's. Such words are called Dyck words in $\{N^n, E^n\}$.

② Triangulations of a convex $(n+2)$-gon with labeled vertices

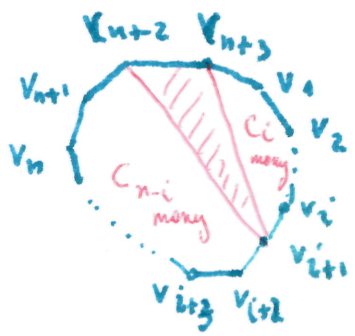
Thm (Euler-Goldbach, Segner) C_n counts # Triangulations of labelled reg $(n+2)$-gon.

Example: $C_3 = 5$



connect 2 Triangulations if related by a single flip & Associahedron

Proof: We show # triangulation satisfies the recurrence: Triang for $n=0$: $\emptyset \rightsquigarrow C_0 = 1$



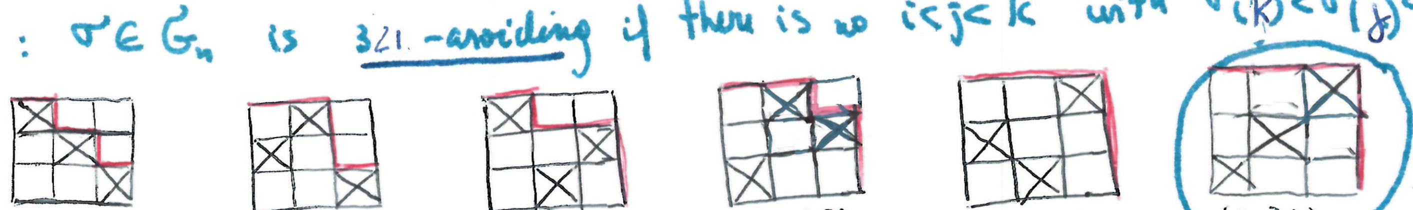
Pick a convex $(n+3)$-gon. To show #Triang $= C_{n+1}$

- Pick $0 \le i \le n$, consider all triangulations where $\{v_{i+1}, v_{n+2}, v_{n+3}\}$ form a Triangle.
- The n-gon is divided into 3 parts
  - an $(n-i+2)$ gon
  - a Δ
  - an $(i+1)$ gon (degenerate if $i=0$)

There are $C_i C_{n-i}$ triangulations for each i, so $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. $\blacksquare$

③ 321-avoiding permutations:

Def: $\sigma \in \mathfrak{S}_n$ is 321-avoiding if there is no $i<j<k$ with $\sigma(k) < \sigma(j) < \sigma(i)$



(1 2 3) (2 1 3) (1 3 2) (3 1 2) (2 3 1) (3 2 1) AVOID THIS

In bijection with any 3-term pattern avoidance

$n=3 \to 5 = C_3$

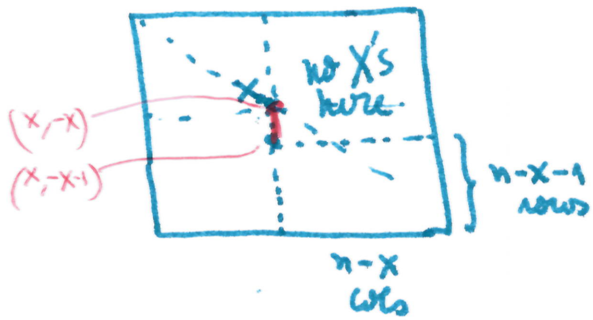
Claim 1: Each permutation gives a lattice path from $(0,0)$ to $(n,-n)$ using E & S steps

We define a map $\phi : \mathfrak{S}_n \longrightarrow \{$Paths from $(0,0)$ to $(n,-n)$ with E & S steps above $x=-y$ line$\}$

$\sigma \longmapsto$ bends on $n\times n$ grid at X's closest to top right.
(upper envelope of the configuration of X's.)

<u>Claim 1</u>: $\phi(\sigma)$ does not cross the $x=-y$ line.

Pf/ If it did, pick $(x,-x)$ first point where we cross, so next step is S



We need to put $(n-x)$ X's in bottom right block, with one X per row.
We don't have enough room!

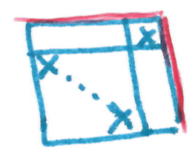
<u>Prop</u>: There is a unique (321)-avoiding permutation for each path in $\operatorname{Im}\phi$.

<u>Corollary</u> $\#\{(321)$-avoiding perm. in $\mathfrak{S}_n\} = \#\{$paths above diagonal on $n\times n$ grid$\} = \#$ Dyck paths $= C_n$.

<u>Proof (Prop)</u>: We proceed by induction on n

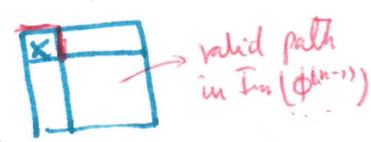
- <u>$n=0,1,2$</u> nothing to check regarding avoidance & both sets have the same size.
- <u>$n=3$</u> By hand
- <u>Inductive Step</u>: Look at the location of X in 1^{st} row (from the Top).

- If spot $=(1,n)$ [top right corner], then $\sigma(n)=1$. The only way to avoid (321) is to have $\sigma(i)<\sigma(j)$ whenever $1\le i<j\le n-1$



<u>Conclude</u> $\sigma(i)=i+1$ $\forall i=1,\dots,n-1$ & $\sigma(n)=1$

- Next, assume spot $=(1,i)$ with $i<n$.

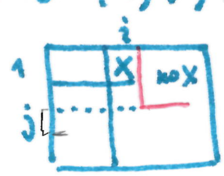


<u>CASE 1</u>: If $i=1$, then $\sigma(1)=1$ & $\sigma\in\mathfrak{S}_{n-1}$

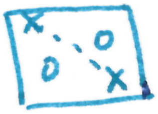
By induction only one (321)-avoiding perm in $\mathfrak{S}_{n-1}$ for this path. Viewed in $\mathfrak{S}_n$, it's also (321)-avoiding.

<u>CASE 2</u>: If $1<i<n$. , know the path bends at $(1,i)$.

Fix the next bending place above row $j\geq 2$



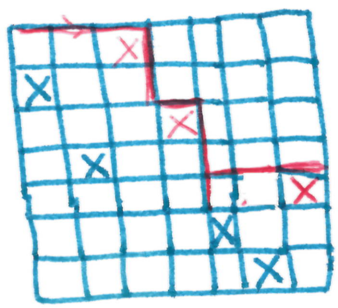
- Since the path doesn't cross the diagonal, then $j \le i$
- The x's in rows $2, \dots, j-1$ must be in columns $1, \dots, i-1$

But if σ is (321)-avoiding, we have no choice but to place them in the first $(j-2)$ columns & in  form. $\Rightarrow$ we know $\sigma|_{1,\dots,j-2,i}$

Now, delete • cols $1, \dots, j-2$ & i
• rows $1, \dots, j-1$
from the grid. This new grid inherits a path above its diagonal & has size $(n-j+1) < n$. By induction, we have $\exists!\ \tilde{\sigma} \in S_{n-j}$ (321)-avoidance corresponding to this path.

Note: $\tilde{\sigma}$ lifts to S_n in a ! way to σ compatible with our prior assignments & σ is (321)-avoiding by construction. □

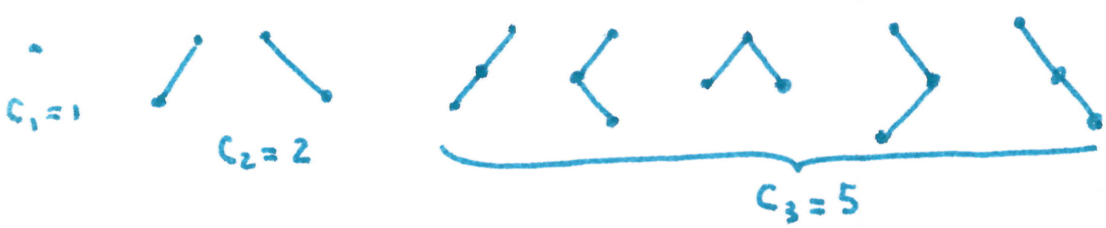
Example:



- path fixes 3 x
- $i = 3$
- $j = 3$ $\Rightarrow$ fixes cross in $(2,1)$
- Next, place $(2,4)$
- Finish with 3×3 block in bottom right corner.

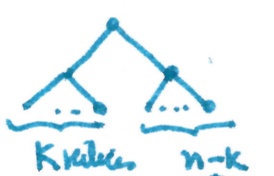
④ Rooted binary plane trees with n vertices = bracketings

- Ex:



Prop: # Bracketings on n vertices $= C_n$

Proof: Pick root for Tree with $(n+1)$ vertices



$\Rightarrow$ Bracketings satisfy the recursion: $(B_n)_n$

$$B_{n+1} = \sum_{k=0}^{n} B_k B_{n-k}$$

$B_0 = 1$ by convention ($B_1 = 1$ ✓)

$\Rightarrow$ (B_n) & (C_n) have same recurrence & initial conditions $\Rightarrow$ they agree! □

More details: EC2 Chapter 6.