Lecture XXIV: Exponential generating Functions, Derangements & Involutions

§1 Convolutions:

OGF's: $\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{m\geq 0} b_m x^m\right) = \sum_{n\geq 0}\left(\sum_{j=0}^{n} a_j b_{n-j}\right) x^n$

The coefficients of the product arises by concatenating objects of type $(a_n)_n$ & (b_n)

Note: Can use this to derive OGF of a product from known OGF of factors.

Ex.: $s_n = \sum_{k=0}^{n} \binom{2k}{k} 4^{-k} = 4^{-n} \sum_{k=0}^{n} \underbrace{\binom{2k}{k}}_{a_k} \underbrace{4^{n-k}}_{b_{n-k}}$

$\sum_{n\geq 0} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}$ & $\sum_{n\geq 0} 4^n z^n = \frac{1}{1-4z}$

$\sum_{n\geq 0} s_n z^n = \frac{1}{(1-4z)^{3/2}}$

$\overset{\text{Binom Series}}{=} \sum_{k\geq 0} \binom{-3/2}{k} (-4)^k z^k$

$\Rightarrow s_n = \binom{-3/2}{n}(-4)^n$

$= \frac{1}{4^n}(2n+1)\binom{2n}{n}$

EGF's: $\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right)\left(\sum_{m\geq 0} b_m \frac{x^m}{m!}\right) = \sum_{n\geq 0}\left(\sum_{j=0}^{n} \frac{a_j}{j!}\frac{b_{n-j}}{(n-j)!}\right) x^n$

$= \sum_{n\geq 0}\left(\sum_{j=0}^{n}\binom{n}{j} a_j b_{n-j}\right)\frac{x^n}{n!}$

The factor $\binom{n}{j}$ on $a_j b_{n-j}$ indicates we are No longer concatenating! Instead, we decompose an n-set into two parts in all possible ways, and for each way we assign a type $(a_n)_{n\geq 0}$ structure and a type $(b_n)_{n\geq 0}$ structure.

Next, we look at 2 examples:

§2. Derangements:

Def. A permutation $\sigma \in \mathfrak{S}_n$ is a derangement if $\sigma(i)\neq i$ $\forall i$ ("σ has no fixed pts")

[Equivalently: If $\sigma = c_1 \cdots c_k$ is the disjoint cycle decomposition, then $|c_i|\geq 2$ $\forall i$]

Example: · $24531 = (12435) \in \mathfrak{S}_5$
· $34152 = (13)(245) \in \mathfrak{S}_5$

Using PIE (Lecture 9), we found $D_n = \#\{\text{derangements in } \mathfrak{S}_n\} = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} k!$

Write $D(x) := \sum_{n\geq 0} D_n \frac{x^n}{n!}$ E.G.F.

Theorem: ① $D(x) = e^{-x}\dfrac{1}{1-x} = \dfrac{e^{-x}}{1-x}$.

② $n! = \displaystyle\sum_{k=0}^{n} \binom{n}{k} D_{n-k}$.

③ $D_n = n! \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

Proof: ③ follows from ① :

$$\sum_{n\geq 0} \underbrace{D_n}\frac{x^n}{n!} = D(x) = e^{-x}\frac{1}{1-x} = \left(\sum_{n\geq 0}(-1)^n\frac{x^n}{n!}\right)\left(\sum_{j\geq 0} x^j\right)$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\cdot 1\right)x^n = \sum_{n\geq 0}\underbrace{\left(n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)}\frac{x^n}{n!}$$

needs to introduce this

Moreover ② & ① are equivalent!

$D(x) \stackrel{?}{=} \dfrac{e^{-x}}{1-x}$ if and only if $e^{x}D(x) \stackrel{?}{=} \dfrac{1}{1-x}$ if and only if

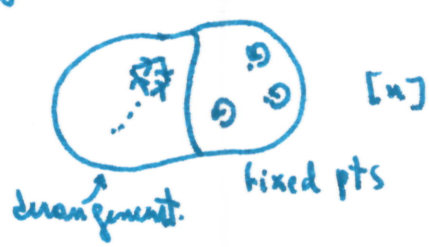
$$\left(\sum_{n\geq 0}\frac{x^n}{n!}\right)\left(\sum_{m\geq 0} D_m\frac{x^m}{m!}\right) \stackrel{?}{=} \sum_{k\geq 0} x^k = \sum_{k\geq 0} k!\frac{x^k}{k!} \quad \text{if and only if}$$

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{1}{k!}\frac{D_{n-k}}{(n-k)!}\right)x^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}D_{n-k}\right)\frac{x^n}{n!} \stackrel{?}{=} \sum_{n\geq 0} n!\frac{x^n}{n!}$$

if and only if $\displaystyle\sum_{k=0}^{n}\binom{n}{k}D_{n-k} = n!$ for all $n\geq 0$.

In short: e.g.f encodes the identity ②.

- Why is ② true? Given $\sigma \in \mathfrak{S}_n$, write σ as the identity on its fixed pts (k many) and a derangement on the non-fixed points ($n-k$ many)



$$\underset{|\mathfrak{S}_n|}{\overset{\shortparallel}{n!}} = \sum_{k=0}^{n} \underbrace{\boxed{\binom{n}{k}}}_{\substack{\text{choose}\\ k \text{ fixed}\\ \text{points}}} \cdot \underbrace{1}_{\substack{\#\text{ id perm}\\ \text{on fixed}\\ \text{pts}}} \cdot \underbrace{D_{n-k}}_{\substack{\#\text{ derangements}\\ \text{on } (n-k) \text{ pts}}}$$

$\square$

§2 Involutions:

Def: A permutation $\sigma \in \mathfrak{S}_n$ is an *involution* if $\sigma^2 = id$

Equivalently: If $\sigma = C_1 \ldots C_k$ is the disjoint cycle decomposition, then $|C_i| \leq 2$ $\forall i$.

Write $\mathrm{invol}_n = \#\{\text{involutions in } \mathfrak{S}_n\}$

Theorem: ① $\sum_{n\geq 0} \text{invol}_n \frac{x^n}{n!} = e^x \cdot e^{\frac{1}{2}x^2} = e^{x+\frac{x^2}{2}}$

② $\text{invol}_n = \sum_{j=0}^{n} \binom{n}{j} (n-j-1)!!$

where: $(2m)!! := 0$

$(2m-1)!! := (2m-1)(2m-3)\cdots 5.3.1$ (product of odd positive numbers $(2m-1)$ & lower)

Proof: Use involution trick to show ② ⇒ ① SUBTLE

$$e^x e^{\frac{x^2}{2}} = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{m\geq 0} \frac{1}{2^m}\frac{x^{\boxed{2m}}}{m!}\right) = \sum_{n\geq 0}\Big(\sum_{\underbrace{1+\cdots+1}_{j}+\underbrace{2+\cdots+2}_{\boxed{m}}=n} \frac{1}{j!}\frac{1}{2^m}\frac{1}{m!}\Big)x^n$$

$$= \sum_{n\geq 0}\Big(\sum_{\substack{j,m\geq 0\\ j+2m=n}} \frac{1}{j!}\frac{1}{2^m}\frac{1}{m!}(j+2m)!\Big)\frac{x^n}{n!}$$

$$= \sum_{n\geq 0}\Big(\sum_{\substack{j,m\geq 0\\ j+2m=n}} \frac{1}{j!}\frac{1}{\boxed{2\cdot 4\cdots(2m-2)(2m)}}(j+2m)!\Big)\frac{x^n}{n!} \qquad = 2^m m!$$

multiply & divide by the missing odd #'s

$$= \sum_{n\geq 0}\Big(\underbrace{\sum_{\substack{j,m\geq 0\\ j+2m=n}} \binom{j+2m}{j} 1.3.5\cdots(2m-1)}_{=\ \text{invol}_n \text{ by ②}}\Big)\frac{x^n}{n!} \qquad [2m = n-j \text{ even}]$$

- To finish, we must prove ②:

Break involution σ into • j one cycles (ie j fixed pts of involution)

• rest = 2-cycles ⇝ $\frac{(n-j)}{2}$ cycles of length 2

[claim: $(n-j-1)!!$ many] (*).

So $\text{invol}_n = \sum_{j=0}^{n} \binom{n}{j} \; 1 \; (n-j-1)!!$

choose fixed pts; id perm on fixed pts; choice of $\frac{(n-j)}{2}$ disjoint 2 cycles among $(n-j)$ elements. ■

Lemma: $\mathcal{P}_n$ = set of partitions of $[n]$ into exactly $\frac{n}{2}$ pairs.

Then $p_n := |\mathcal{P}_n| = (n-1)!!$

Pf/ If n odd, we can't have a partition into pairs. $p_n = 0$ & $(n-1)!! = 0$ also.

If n is even, $n = 2j$ then $p_{2j} = \underbrace{(2j-1)}\, p_{2j-2}$

↳ pick the element paired with $2j = n$

Induction j ⇒ $p_{2j} = (2j-1)!!$ □

§3 Summary:

Remark: For both derangements & involutions, we use a product of egf's to get an identity involving choosing a subset of $[n]$ and placing structure on the subset and it's complement.

Now, we generalize this:

Fix M a type of combinatorial structure on labeled finite sets, and let

- $m_n :=$ number of such structures on a labeled set of size n. for $n \geq 1$
- We extend to m_0, typically with value 0 or 1.

Def: $M(x) := \sum_{n\geq 0} m_n \frac{x^n}{n!}$ to be the egf of type M.

Examples: ① M = permutations, so $m_n = |\mathfrak{S}_n| = n!$ and $m_0 = 1$.

So $M(x) = \sum_{n\geq 0} m_n \frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x}$.

② M = trivial structure (on a set), so $m_n = 1$ & $m_0 = 1$

Then $M(x) = \sum_{n\geq 0} m_n \frac{x^n}{n!} = \sum_{n\geq 0} 1 \cdot \frac{x^n}{n!} = e^x$.

③ P = set of partitions of $[n]$ into exactly $\frac{n}{2}$ pairs

Lemma gives $P_{2j+1} = 0 \; \forall j$ & $P_{2j} = (2j-1)!!$ $\leadsto$ $P_n = (n-1)!!$

Then: $M(x) = \sum_{j\geq 0} (2j-1)!! \frac{x^{2j}}{(2j)!} = \sum_{j\geq 0} \frac{x^{2j}}{2^j \cdot j!} = e^{\frac{x^2}{2}}$.

Pairwise-Compound Structure Principle:

To obtain egf for a combinatorial structure M formed as a pairwise-compound structure of 2 other structures of type A & B, where each structure of M decomposes uniquely into 2 structures of type A & B, resp., we compute:

$$m_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}$$

Equivalently, $M(x) = A(x)\,B(x)$

Proof: Enough to prove the statement about m_n.

The (RHS) records the unique decomposition, ie.

$$\sum_{j=0}^{n} \binom{n}{j} \overbrace{a_j \quad b_{n-j}}^{\text{decomp for } j \text{ fixed.}} = m_n \checkmark$$

$\boxed{!\ \text{decomp.}}$

$\boxed{\text{choose } j \text{ elements from the } n\text{-set}}$

# of Type A structures on the j-elements

# of Type B structures on the $(n-j)$ other elements

$\square$

Obs: $e^x D(x) = \frac{1}{1-x}$ & $\sum_{n\geq 0} \mathrm{invol}_n \frac{x^n}{n!} = e^x e^{x^2/2}$ both follow from this principle.