

Lecture XXV: Basics in Polyhedral Geometry

GOALS: • Basic definitions via examples

• Show how to use discrete data to compute geometric invariants.

For Volume \leadsto Ehrhart Theory

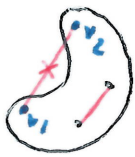
§1 Basics in Convex Geometry

Fix V a finite dimensional vector space / \mathbb{R} . Later on: $V \cong \mathbb{R}^n$ by choosing a basis of V .

Def: $C \subseteq V$ is called convex if for every $v_1, v_2 \in C$ we have:

$$[v_1, v_2] = \{ \lambda v_1 + (1-\lambda)v_2 : 0 \leq \lambda \leq 1 \} \subseteq C$$

Ex:



not convex

Def Given any $U \subseteq V$ any nonempty subset, we build $\text{Conv}(U) = \text{convex hull of } U$
= smallest convex set in V containing U .

Lemma: Convex hulls exist & are unique.

Proof: • V is convex

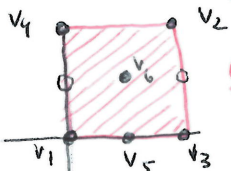
• V_1, V_2 convex $\Rightarrow V_1 \cap V_2$ convex

Similarly $(V_i)_{i \in I}$ convex $\Rightarrow \bigcap_i V_i$ convex

$\Rightarrow \text{Conv}(U) := \bigcap_{\substack{W \supseteq U \\ W \text{ convex}}} W$ is the convex hull of U . \square

Def If U is finite, then $\text{conv}(U)$ is called a polytope.

Ex: $v_1 = (0,0), v_2 = (2,2), v_3 = (2,0), v_4 = (0,2), v_5 = (1,0), v_6 = (1,1)$



$$\text{conv}(\{v_1, \dots, v_6\}) = \square_2$$

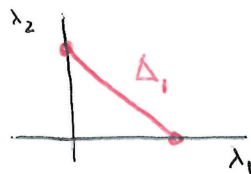
Note $\text{conv}(v_1, v_2, v_3, v_4) = \square_2$ and $\{v_1, \dots, v_4\}$ is minimal with this property
 \Rightarrow Vertices of \mathcal{P} .

Def: Vertices of a polytope = minimal set generating \mathcal{P} as $\text{conv}(U)$.

Consequence: If $V(\mathcal{P}) = \{v_1, \dots, v_m\}$, then $\mathcal{P} = \bigcap_v \{ \lambda_1 v_1 + \dots + \lambda_m v_m \mid \lambda_1, \dots, \lambda_m \geq 0, \lambda_1 + \dots + \lambda_m = 1 \}$

Note = $\{(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \mid \lambda_i \geq 0 \forall i, \sum \lambda_i = 1\}$ is a special polytope in \mathbb{R}^m ,
called the $(m-1)$ -standard simplex. $=: \Delta_{m-1}$

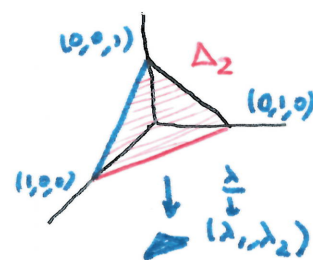
Ex: $m=2$



$$\lambda_1, \lambda_2 \geq 0$$

$$\lambda_1 + \lambda_2 = 1$$

$m=3$



$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

projection
to \mathbb{R}^m



$\Delta_2 \cap (\lambda_3 = 0) = \Delta_1 \times \{0\}$
& some further 2 coordinate hyperplanes

In general: $\Delta_{m-1} \xrightarrow{\psi} \mathbb{R}^{m-1}$
 $(\lambda_1, \dots, \lambda_m) \mapsto (\lambda_1, \dots, \lambda_{m-1})$

$$\psi(\Delta_{m-1}) = \{(\lambda_1, \dots, \lambda_{m-1}) \in \mathbb{R}_{\geq 0}^{m-1} \mid \sum \lambda_i \leq 1\}$$

Consequence: \mathcal{P} is the image of Δ_{m-1} under the linear map.

$$\Delta_{m-1} \xrightarrow{\Phi} \mathcal{P}$$

$$(\lambda_1, \dots, \lambda_m) \mapsto \sum \lambda_i v_i$$

If $V \cong \mathbb{R}^n$, Φ is continuous
 Δ_{m-1} is compact $\Rightarrow \mathcal{P}$ is compact (closed and bounded)

Def: $\dim(\mathcal{P}) :=$ dimension of its affine span $(:= \{x + \lambda(y-x) \mid y \in \mathcal{P}, \lambda \in \mathbb{R}\})$
for any $x \in \mathcal{P}$ fixed.

Ex: $\dim \Delta_{m-1} = (m-1)$.

Def: $C \subseteq V$ is a cone if $v \in C \Rightarrow \lambda v \in C \forall \lambda \geq 0$.

Note: Every nonempty cone contains $0 \in V$.

Examples

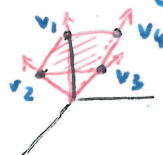
(0) V

(1) Lines through the origin

(2)  cone over circle at height 1.

(3) Our favorite cones:

[polyhedral cone]



$$v_1 = (0, 0, 1)$$

$$v_2 = (1, 0, 0)$$

$$v_3 = (1, 1, 1)$$

$$v_4 = (0, 1, 1)$$

[non-polyhedral]

cone over square  \Rightarrow

Def: A polyhedral cone C is a cone generated by finitely many vectors, i.e.
 $C = C(v_1, \dots, v_m) = \{\lambda_1 v_1 + \dots + \lambda_m v_m \mid \lambda_i \geq 0\} = \mathbb{R}_{\geq 0} \langle v_1, \dots, v_m \rangle$.

Def: A minimal generating set for C is the set of rays of C

(why? a minimal set is unique!)

Consequence: $C \subseteq \mathbb{R}^n$ is closed

2 Special cases & polytopes = generators in \mathbb{Z}^n or \mathbb{Q}^n where $V \cong \mathbb{R}^n$ is fixed

Names: For polytope P : if $V(P) \subseteq \mathbb{Z}^n \Rightarrow P$ is a lattice polytope

$V(P) \subseteq \mathbb{Q}^n \Rightarrow P$ is a rational polytope.

For cones C : $V(C) \subseteq \{\text{rays of } C\} \subseteq \mathbb{Q}^n \Rightarrow C = \text{rational polyhedral cone}$.

§2. V- vs H-representations:

Write $P = \text{conv}\{v_1, \dots, v_m\}$ or $C = \mathbb{R}_{\geq 0} \langle v_1, \dots, v_m \rangle$ for V-representation

H-representation via example:

$$P = \text{conv}\{(0,1), (1,1), (1,0), (0,0)\}$$



H_0, H_1 : 2 examples of supporting hyp.

H_2 not supporting

Want to write P or C as solutions to linear inequalities

\Rightarrow supporting hyperplanes H

Def: $H = \{a_1 x_1 + \dots + a_n x_n = b\}$ is a supporting hyperplane if $H \cap P \neq \emptyset$ & P lies on one of the 2 halfspaces $\begin{cases} a_1 x_1 + \dots + a_n x_n \geq b \\ a_1 x_1 + \dots + a_n x_n \leq b \end{cases}$

Def: If H is supporting hyperplane $\Rightarrow H \cap P$ is a face of P .
(Same for faces of polyhedral cones C) [Example above: 8 faces = 4 vertices + 4 edges]

Lemma: A face of P is again a polytope ($V(F) = V(P) \cap F$)

———— C ———— polyhedral cone ($V(F) = V(C) \cap F$)

We can collect the faces according to their dimension.

Def f-vector(P) = vector in $\mathbb{R}^{\dim P + 1}$ with integer coordinates (f_0, \dots, f_d)

where $f_i := \#$ faces of P of dim i . $d = \dim P$

Example ① f-vector $(\square) = (4, 4, 1)$ $\Rightarrow \chi_P = \sum_{i=0}^{\dim P} (-1)^i f_i$
 $\downarrow \quad \quad \downarrow \quad \quad \downarrow$
 $\dim=2 \quad \text{vertices} \quad \text{edges} \quad \rightarrow P \text{ itself}$
 $= 4 - 4 + 1 = 1$

P is contractible $\Rightarrow \chi_P = \chi_{\text{pt}} \Rightarrow$ it must be 1. (Euler characteristic of P viewed as a cell-complex)

② f-vector $(\triangle) = (1, 2, 1)$
 $\downarrow \quad \quad \downarrow$
 $\dim=2 \quad \text{vertices}$

Supporting hyperplanes yield linear inequalities (finitely many)

$$\Rightarrow C = H_1 \cap \dots \cap H_m$$

$$H_i = \left\{ \sum_j a_j^{(i)} x_j \leq b_i \right\} \quad \Rightarrow A \in \mathbb{R}^{m \times n} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

If C is bounded then it is a polytope.

In general, C is a polyhedron. If C is a cone, we can always choose constants $b_i = 0 \quad \forall i$ (see examples below)

Def: H-representation $(C) = \{ \underline{x} \in \mathbb{R}^n \mid A \cdot \underline{x} \leq \underline{b} \}$

Note: Since $\{ a_1 x_1 + \dots + a_n x_n \geq b \} = \{ (-a_1) x_1 + \dots + (-a_n) x_n \leq (-b) \}$


We can always assume supporting hyperplanes satisfy $C_P \subseteq \{ \tilde{a}_1 x_1 + \dots + \tilde{a}_n x_n \leq \tilde{b} \}$

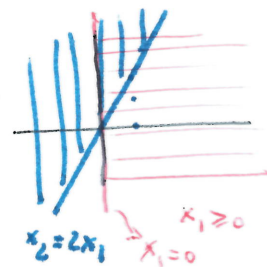
If we eliminate redundant rows of A , the remaining inequalities describe the faces of P (resp. C), i.e. faces of dimension 1 less than the dim of P (resp. C)

Example: ① $A = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} -x_1 &\leq 0 \\ -x_2 + 2x_1 &\leq 0 \end{aligned}$$

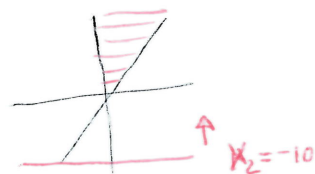
To draw: draw the 2 halfspaces & intersect them.

$$\Rightarrow \{ A \underline{x} \leq \underline{b} \} =$$




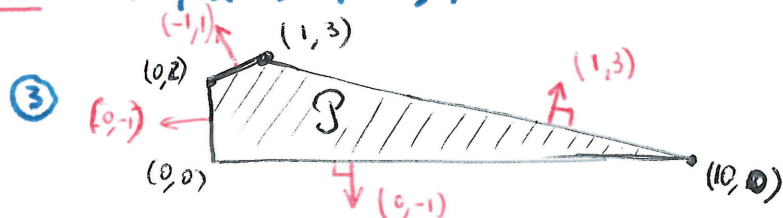
② $A = \begin{pmatrix} -1 & 0 \\ 2 & -1 \\ 0 & -1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$

$$\begin{aligned} -x_1 &\leq 0 \\ -x_2 + 2x_1 &\leq 0 \\ -x_2 &\leq 10 \end{aligned}$$



gives the same as ①.

Note: The fact that $b_3 \neq 0$ is consistent with the statement $-x_2 \leq 10$ is redundant



4 faces supported by $\vec{n}_i \cdot \begin{pmatrix} x \\ y \end{pmatrix} = b_i$

where $\vec{n}_i = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
 $b_i = 0, 10, 2, -2$

$$H\text{-rep}(P) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 3 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ 10 \\ 2 \\ -2 \end{pmatrix} \right\}$$

Fourier-Motzkin algorithm: allows to go between the H- & V-representations of polytopes (see Ziegler's Polytopes book, or [Blek-Sinai])

For cones we also have a similar result:

$$\text{V-representation } (C) = \{ B \cdot \underline{y} \mid \underline{y} \geq 0 \} \quad \text{for } B \in \mathbb{R}^{n \times p} \quad \text{cols}(B) = \text{rays of } C.$$

\downarrow
 $y_i \geq 0$

Thm: For every $A \in \mathbb{R}^{m \times n}$, there exists $B \in \mathbb{R}^{n \times p}$ (for some p , depending on A)

$$\text{such that } \{ \underline{x} \mid A \cdot \underline{x} \leq \underline{0} \} = \{ B \cdot \underline{y} \mid \underline{y} \geq 0 \}$$

Conversely, for every $B \in \mathbb{R}^{n \times p}$, there exists $A \in \mathbb{R}^{m \times n}$ (for some m , depending on B)

$$\text{such that } \{ B \cdot \underline{y} \mid \underline{y} \geq 0 \} = \{ \underline{x} \mid A \cdot \underline{x} \leq \underline{0} \}$$

In short: can go between V- & H-representations of polyhedral cones.