Lecture XXVI: Introduction to Ehrhart Theory

Last time: discussed polytopes & polyhedral cones

Special type: lattice polytopes in $V \simeq \mathbb{R}^n$ = vertices in $\mathbb{Z}^n$

GOAL: Compute volumes discretely ($\sim$ quantum approach)

Replacing $\mathbb{R}^n$ by affine span $(\mathcal{P}) = \{x + \lambda(y-x) \quad \lambda \in \mathbb{R}, y \in \mathcal{P}\}$ for any $x \in \mathcal{P}$

we can assume $\dim(\mathcal{P}) = n$ ($\mathcal{P}$ is full-dimensional)

§1. What is Ehrhart Theory?

Input: $\mathcal{P}$ full-dim'l lattice polytope in $\mathbb{R}^n$ (later: extend to rational case, ie $V(\mathcal{P}) \subseteq \mathbb{Q}^n$)

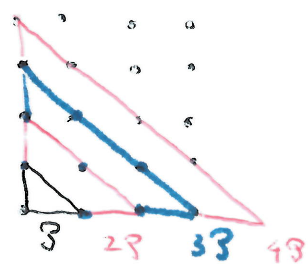
Output: Discrete Volumes of dilations of $\mathcal{P}$

Fix $t \in \mathbb{Z}_{>0}$ & set $t\mathcal{P} :=$ t-fold dilation of $\mathcal{P}$
$= \text{Conv Hull}(\{tv : v \in V(\mathcal{P})\})$

Def: $L_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^n) = \#(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^n)$

($t\mathcal{P}$: dilate polytope; $\frac{1}{t}\mathbb{Z}^n$: rescale lattice)

Example: $\mathcal{P} = $ triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$



t	1	2	3	4
$L_{\mathcal{P}}(t)$	3	6	10	15

$L_{\mathcal{P}}(0) = 1$ by def.

$$L_{\mathcal{P}}(t) - L_{\mathcal{P}}(t-1) = \#\{(x,y) \mid x,y \geq 0 \quad x+y = t\} = t+1 \quad \forall t \geq 1$$

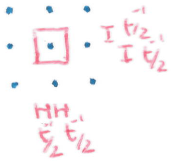
So Telescopic sum gives $L_{\mathcal{P}}(t) - \underbrace{L_{\mathcal{P}}(0)}_{=1} = \sum_{j=1}^{t} \underbrace{(L_{\mathcal{P}}(j) - L_{\mathcal{P}}(j-1))}_{j+1} = \sum_{j=2}^{t+1} j$

$$\Rightarrow \boxed{L_{\mathcal{P}}(t)} = \sum_{j=1}^{t+1} j = \frac{(t+2)(t+1)}{2} = \boxed{\binom{t+2}{2} \quad \forall t \geq 1}.$$

$L_{\mathcal{P}}(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1$ & $\frac{L_{\mathcal{P}}(t)}{t^2} \xrightarrow[t \to \infty]{} \frac{1}{2} = \text{Vol}(\mathcal{P})$.

Prop: $\mathrm{Vol}(\mathcal{P}) = \lim_{t\to\infty} \frac{L_{\mathcal{P}}(t)}{t^n}$ $n = \dim(\mathcal{P})$.

Proof: Approximate $\mathcal{P}$ by unit cubes in $\frac{1}{t}\mathbb{Z}^n$ with center of mass at points in $\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^n$

(ie  & higher dim'l analogs)

Then $\mathrm{vol}_{\mathcal{P}} = \int_{P} 1\, d\underline{z} = \lim_{t\to\infty} L_{\mathcal{P}}(t) \underbrace{\mathrm{Vol}(\text{cubes})}_{= (\frac{1}{t})^n} = \lim_{t\to\infty} \frac{L_{\mathcal{P}}(t)}{t^n}$ □

(Riemann sums; $L_{\mathcal{P}}(t)$ = # cubes)

• **Thm 1** (Ehrhart 1962) $\mathcal{P}$ full-dim'l lattice polytope in $\mathbb{R}^n$

Then: $L_{\mathcal{P}}(t) \in \mathbb{Q}[t]$ of degree $n = \dim \mathcal{P}$, ie

$$L_{\mathcal{P}}(t) = \underbrace{L_n(\mathcal{P})}_{\in\mathbb{Q}} t^n + \underbrace{L_{n-1}(\mathcal{P})}_{\in\mathbb{Q}} t^{n-1} + \cdots + \underbrace{L_0(\mathcal{P})}_{\in\mathbb{Q}} \quad \forall t \in \mathbb{Z}_{>0}$$

Name: $L_{\mathcal{P}}(t)$ = Ehrhart polynomial.

• Prop ⇒ $L_n(\mathcal{P}) = \mathrm{Vol}(\mathcal{P})$.

Example: $L_\Delta(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1$ values at $t>0$ match up with computations

⇝ Can use $\dim\mathcal{P}+1$ values of $t_{>0}$ to compute $L_{\mathcal{P}}(t)$

Q: Can we count $\mathbb{Z}$-pts in $\mathrm{int}(\mathcal{P})$? What about dilations?

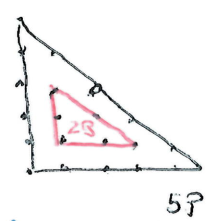
A: YES!

$$\mathrm{int}(\mathcal{P}) = \mathcal{P} \smallsetminus \partial\mathcal{P} = \mathcal{P} \smallsetminus \bigcup_{F \text{ facet of } \mathcal{P}} F$$

Def $L_{\mathrm{int}(\mathcal{P})}(t) = \#\{\mathrm{int}(\mathcal{P}) \cap \frac{1}{t}\mathbb{Z}^n\}$

Example: $\mathcal{P} = \Delta$

t	1	2	3	4	5
$L_{\mathrm{int}(\mathcal{P})}(t)$	0	0	1	3	6



$$L_{\mathrm{int}(\mathcal{P})}(t) = L_{\mathcal{P}}(t-3) = \frac{(t-3+2)(t-3+1)}{2} = \frac{(t-1)(t-2)}{2} = (-1)^2 \frac{(-t+1)(-t+2)}{2} = L_{\mathcal{P}}(-t)$$

• **Thm 2** (Ehrhart-Macdonald Reciprocity) $\mathcal{P} \subseteq \mathbb{R}^n$ full-dim'l lattice polytope.

$$L_{\mathrm{int}\,\mathcal{P}}(t) = (-1)^n L_{\mathcal{P}}(-t).$$

Ehrhart Series: $\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t)\, z^t = \sum_{t\geq 0} L_{\mathcal{P}}(t)\, z^t$ when $L_{\mathcal{P}}(0) := 1$.

Thm 3 (Ehrhart): $\mathrm{Ehr}_{\mathcal{P}}(z)$ is a rational function for $\mathcal{P} \subseteq \mathbb{R}^n$ full-dim'l lattice polytope. Furthermore, it has the form:

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{j=0}^{n} h_j^*(\mathcal{P})\, z^j}{(1-z)^{n+1}} \quad \text{with} \quad \sum_{j=0}^{n} h_j^*(\mathcal{P}) \neq 0.$$

($z=1$ is a pole of order $(n+1)$ of $\mathrm{Ehr}_{\mathcal{P}}$)

Name: $(h_0^*, h_1^*, \ldots, h_n^*) = h^*\text{-vector}(\mathcal{P})$.

Thm 4 (Stanley's Non-Negativity) $h_0^*, h_1^*, \ldots, h_n^* \in \mathbb{Z}_{\geq 0}$. ($\mathcal{P}$ = full dim'l lattice polytope in $\mathbb{R}^n$)

Remark: If we work with rational polytopes ($V(\mathcal{P}) \subseteq \mathbb{Q}^n$), we obtain slightly different behavior: instead of $L_{\mathcal{P}} \in \mathbb{Q}[t]$ we will have a quasipolynomial (coefficients = periodic functions with integer period, eg $(-1)^t$ vs 1 or -1)

Example: $\mathrm{Ehr}_{\Delta}(z) = 1 + \sum_{t\geq 1} \frac{(t+2)(t+1)}{2} z^t$

Integrate Twice:

$$\int \mathrm{Ehr}_{\Delta}(z)\, dz = z + \sum_{t\geq 1} \frac{(t+2)}{2} z^{t+1} + C$$

$$\int\left(\int \mathrm{Ehr}_{\Delta}(z)\, dz\right) dz = \frac{z^2}{2} + \sum_{t\geq 1} \frac{z^{(t+2)}}{2} + Cz + D$$

$$= \sum_{t\geq 0} \frac{z^{t+2}}{2} + Cz + D = \frac{1}{2}\left(\frac{1}{1-z} - 1 - z\right) + Cz + D$$

$\Rightarrow$ Differentiate Twice to get $\mathrm{Ehr}_{\Delta}(z)$:

$$\mathrm{Ehr}_{\Delta}(z) = \left(\frac{1}{2}\frac{1}{1-z}\right)'' = \left(\frac{1}{2(1-z)^2}\right)' = \frac{1}{(1-z)^3}$$

so $h^*(\Delta) = (1,0,0)$ $\quad |h^*| = \sum_{j=0}^{n} h_j^* = 1 \neq 0$.

§2: The standard simplex

$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n \leq 1,\ x_i \geq 0\ \forall i\} \subseteq \mathbb{R}^n$ full dim'l

$t\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n \leq t,\ x_i \geq 0\ \forall i\}$ for $t \in \mathbb{Z}_{>0}$

⇝ Add slack variable $x_{n+1} \geq 0$ & require $x_1 + \cdots + x_n + x_{n+1} = t$

Since we want $t\Delta_n \cap \mathbb{Z}^n$, $x_1, \ldots x_n, x_{n+1} \in \mathbb{Z}_{\geq 0}$ $\Longrightarrow$ count weak $(n+1)$-compositions of t.

Lecture 3: $\#(t\Delta_n \cap \mathbb{Z}^n) = \binom{t+(n+1)-1}{(n+1)-1} = \binom{t+n}{n}$ (matches our $n=2$ example)

$\Rightarrow$ $\boxed{L_{\Delta_n}(t) = \binom{t+n}{n}}$ degree n polynomial with $\mathbb{Q}$-coeffs.

$vol(\Delta_n) = \frac{1}{n!} = \lim_{t \to \infty} \frac{\binom{t+n}{n}}{t^n}$.

. Interior pt count?

$$L_{int(\Delta_n)}(t) = \#\{(x_1, \ldots, x_n) \in \mathbb{Z}^n_{>0} \mid x_1 + \cdots + x_n < t\}$$

$$= \#\{(m_1, \ldots, m_{n+1}) \in \mathbb{Z}^n_{>0} \mid m_1 + \cdots + m_{n+1} = t\}$$

$$= \#((n+1)\text{-compositions of } t)$$

$$= \binom{t-1}{n+1-1} = \binom{t-1}{n}$$

Obs: $\binom{-t+n}{n} = \frac{(-t+n)(-t+(n-1))\cdots(-t+1)}{n!} = (-1)^n \frac{(t-n)(t-(n-1))\cdots(t-1)}{n!}$

$= (-1)^n \binom{t-1}{n}$

So $L_{int(\Delta_n)}(t) = (-1)^n L_{\Delta_t}(-t)$ as we wanted.

$$Eh_{\Delta_n}(z) = 1 + \sum_{t \geq 1} \binom{t+n}{n} z^t = \sum_{t \geq 0} \binom{t+n}{n} z^t \underset{\uparrow}{=} \frac{1}{(1-z)^{n+1}}.$$

↑ Binomial Series