Lecture XXVII : Pick's Theorem & Ehrhart polynomials

Last Time: 4 Theorems in Ehrhart Theory of lattice, full-dim'l polytopes in $\mathbb{R}^n$

- $L_P(t) := \#(tP \cap \mathbb{Z}^n) = \#(P \cap \frac{1}{t}\mathbb{Z}^n)$ for $t \in \mathbb{Z}_{\geq 1}$.
- $L_{P^\circ}(t) := \#((tP)^\circ \cap \mathbb{Z}^n) = \#(P^\circ \cap \frac{1}{t}\mathbb{Z}^n)$

1) Ehrhart: $L_P(t) \in \mathbb{Q}[t]$ of degree n [Ehrhart] & $L_P(t) = \sum_{j=0}^{n} L_j(P)\, t^j$, $L_n(P) = Vol(P)$

2) Ehrhart-Macdonald Reciprocity $L_{P^\circ}(t) = (-1)^n L_P(-t)$

(This gives a geometric interpretation for evaluating L_P at $\mathbb{Z}_{<0}$.)

3) Ehrhart Series: $Ehr_P(z) := 1 + \sum_{t \geq 1} L_P(t)\, z^t$ is a rational function in z, of the form

$$Ehr_P(z) = \frac{\sum_{j=0}^{n} h_j^*(P)\, z^j}{(1-z)^{n+1}} \qquad h^*(P) = (h_0^*(P), \ldots, h_n^*(P))$$

In particular, $h_0^* = 1$. $\qquad |h^*(P)| = \sum_{j=0}^{n} h_j^*(P) \neq 0$

4) Stanley's non-Negativity: $h^*(P) \in \mathbb{Z}_{\geq 0}^{n+1}$.

($h_j^*(P)$ will count something)

Roadmap: 3 $\Rightarrow$ 1 & 2 ; Proof technique of 3 $\Rightarrow$ 4.

§1: Proof of $\exists$ Ehrhart polynomial (3 $\Rightarrow$ 1)

Lemma: Assuming the Ehrhart Series has the claimed rational expression, then

$$L_P(t) = \binom{t+n}{n} + h_1^*\binom{t+n-1}{n} + \cdots + h_{n-1}^*\binom{t+1}{n} + h_n^*\binom{t}{n}$$

Proof:

$$Ehr_P(z) = (h_n^* z^n + h_{n-1}^* z^{n-1} + \cdots + h_1^* z + 1)\frac{1}{(1-z)^{n+1}}$$

$$\overset{\boxed{\text{Binomial Series}}}{=} (h_n^* z^n + \cdots + h_1^* z + 1)\Big(\sum_{t \geq 0} \binom{t+n}{n} z^t\Big)$$

$$= h_n^* \sum_{t \geq 0} \binom{t+n}{n} z^{n+t} + h_{n-1}^* \sum_{t \geq 0} \binom{t+n}{n} z^{t+n-1} + \cdots + h_1^* \sum_{t \geq 0} \binom{t+n}{n} z^{t+1} + \sum_{t \geq 0} \binom{t+n}{n} z^t$$

$$= h_n^* \sum_{t \geq n} \binom{t}{n} z^t + h_{n-1}^* \sum_{t \geq n-1} \binom{t+1}{n} z^t + \cdots + h_1^* \sum_{t \geq 1} \binom{t+n-1}{n} z^t + \sum_{t \geq 0} \binom{t+n}{n} z^t$$

Extend all sums to $t \geq 0$ since binomial coeffs $= 0$

$$= \sum_{t \geq 0} \underbrace{\Big(h_n^* \binom{t}{n} + h_{n-1}^* \binom{t+1}{n} + \cdots + h_1^* \binom{t+n-1}{n} + \binom{t+n}{n}\Big)}_{=: L_P(t) \ [t \geq 0]} z^t. \quad \blacksquare$$

Consequence 1: $L_{\mathcal{P}}(0) := 1$ (ie evaluation of Ehrhart polynomial at 0)

Consequence 2: In the basis (needs a proof!) $\{\underset{\deg n}{\binom{t}{n}}, \underset{\deg n}{\binom{t+1}{n}}, \ldots, \underset{\deg n}{\binom{t+n}{n}}\}$ of $\mathbb{Q}[t]_{\leq n}$,

$\hookrightarrow$ polys of $\mathbb{Q}[t]$ of degree $\leq n$ $\cup \{0\}$

$L_{\mathcal{P}}(t)$ has explicit coefficients ⇝ h^*-vector of $\mathcal{P}$.
[all non-neg integers by Stanley's non-neg]

Corollary: $h_1^*(\mathcal{P}) = L_{\mathcal{P}}(1) - (n+1) = \#(\mathcal{P} \cap \mathbb{Z}^n) - n - 1$.

Proof: Set $t = 1$: $\binom{1+n-2}{n} = \binom{1+n-3}{n} = \cdots = \binom{1+1}{n} = \binom{1}{n} = 0$ for n

$\Rightarrow L_{\mathcal{P}}(1) = \binom{1+n}{n} + h_1^* \binom{1+n-1}{n} = n+1 + h_1^* \cdot 1$. ■

- $h_2^*, \ldots, h_n^*$ have similar interpretations.

Assuming Stanley's non-negativity, we obtain a bound on denominators of $L_{\mathcal{P}}(t)$

Corollary: If $L_{\mathcal{P}}(t) = c_n t^n + \cdots + c_1 t + 1$, then $n!\, c_k \in \mathbb{Z}$ $\forall k = 1, \ldots, n$.

Pf/ Use $L_{\mathcal{P}}(t) = \sum_{j=0}^{n} \underset{\mathbb{Z}}{\underset{\cap}{h_j^*}} \binom{t+n-j}{n}$ ⇝ $n!$ appears as denominators.

§2. Pick's Theorem:

GOAL: Compute $L_{\mathcal{P}}(t)$, $L_{\mathcal{P}^\circ}(t)$, $\mathrm{Ehr}_{\mathcal{P}}(z)$ for $\mathcal{P} \subseteq \mathbb{R}^2$ full-dimensional, lattice polygon.

Theorem (Pick's): $\mathcal{P}$ integer polygon: $\mathrm{Area}(\mathcal{P}) = I + \frac{1}{2} B - 1$, where

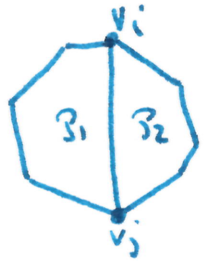
$I = L_{\mathcal{P}^\circ}(1) = \#(\mathrm{int}(\mathcal{P}) \cap \mathbb{Z}^2)$

$B = L_{\mathcal{P}}(1) \smallsetminus L_{\mathcal{P}^\circ}(1) = \#(\partial \mathcal{P} \cap \mathbb{Z}^2)$.

Trick: "divide and conquer" via Triangulation of $\mathcal{P}$ & show additivity of the formula under subdivisions.

Lemma 0: Decompose $\mathcal{P}$ by a chord joining 2 non-adjacent vertices & write $\mathcal{P}_1$ & $\mathcal{P}_2$ for the resulting polygons in the subdivision of $\mathcal{P}$.
If Pick's Thm holds for $\mathcal{P}_1$ & $\mathcal{P}_2$ $\Rightarrow$ holds for $\mathcal{P}$. | Also, if holds for $\mathcal{P}$ & $\mathcal{P}_1$ $\Rightarrow$ holds for $\mathcal{P}_2$

Proof:



$A = \mathrm{Area}(\mathcal{P})$
$A_1 = \mathrm{Area}(\mathcal{P}_1)$
$A_2 = \mathrm{Area}(\mathcal{P}_2)$
$L = \#([v_i, v_j] \cap \mathbb{Z}^2)$

$A = A_1 + A_2$

$I = I_1 + I_2 + L - 2$ (counted v_i, v_j twice!)

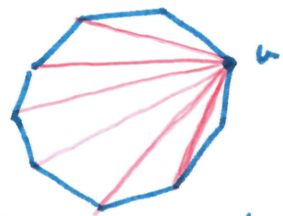
$B = B_1 + B_2 - 2L + 2$ (counted L in B_1 & B_2)

(1) If $\left.\begin{array}{l} A_1 = I_1 + \frac{1}{2}B_1 - 1 \\ A_2 = I_2 + \frac{1}{2}B_2 - 1 \end{array}\right\} \Rightarrow I + \frac{B}{2} - 1 = I_1 + I_2 + L - 2 + \frac{B_1}{2} + \frac{B_2}{2} - L + 1 - 1$

$$= \left(I_1 + \frac{B_1}{2} - 1\right) + \left(I_2 + \frac{B_2}{2} - 1\right) = A_1 + A_2 = A$$

(2) Same calculation: If $\left.\begin{array}{l} A_1 = I_1 + \frac{1}{2}B_1 - 1 \\ A = I + \frac{1}{2}B - 1 \end{array}\right\} \Rightarrow I_2 + \frac{B_2}{2} - 1 = A_2$. □

Consequence: By induction on the number of sides of P, it's enough to prove it for triangles, since we can always triangulate P by drawing chords from one fixed vertex



• Proof Strategy for triangles with integer vertices (= lattice triangle)

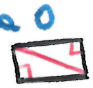
Lemma 1: Any lattice triangle T lies in a rectangle with sides parallel to x- and y-axis. Furthermore, we can subdivide the enclosing rectangle into rectangles with the same property, T and special lattice triangles: right triangles with 2 sides parallel to the axes.

Up to symmetry, there are 3 cases:

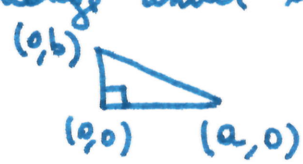
[label vertices in increasing x-coords & look at the heights: UU or UD.
up to $y \to -y$ or $x \to -x$ symmetries]

• By Lemma 0, Pick's Theorem will follow for any lattice triangle if we prove it for  & the special triangles 

• Further symmetry in $\mathbb{Z}^2$ & Lemma 0 again: enough to prove it for .



• Since A, I & B don't change under translations in $\mathbb{Z}^2$, we need only prove Pick's Theorem for



$\rightsquigarrow \boxed{\begin{array}{l} 0 \leq x \leq a \\ 0 \leq y \leq b \\ y \leq -\frac{b}{a}x + b \end{array}}$

Proof: Next Time