# Lecture XXVIII: Pick's Theorem, Ehrhart Theory in dim 2 & for simplicial polytopes

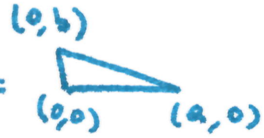
## §1 Pick's Theorem:

**Pick's Thm**: Fix $P \subseteq \mathbb{R}^2$ lattice polygon. Then $A(P) = I(P) + \frac{B(P)}{2} - 1$,

where $A(P)$ = area of $P$

$I(P) = \#(P^\circ \cap \mathbb{Z}^2)$

$B(P) = \#(\partial P \cap \mathbb{Z}^2)$

**Last time**: reduced to proving one case: $P =$ 

$a, b \in \mathbb{Z}_{>0}$.

**Lemma**: Pick's Thm holds for this special triangle.

Pf/ Note $P = \begin{cases} 0 \leq x \leq a \\ 0 \leq y \leq b \\ -\frac{b}{a}x + y \leq b \end{cases}$ $\quad \boxed{A(P) = \frac{ab}{2}}$

$$B(P) = \underset{-}{(a+1)} + \underset{|}{(b+1)} + \underbrace{\#(\diagdown \cap \mathbb{Z}^2)}_{=:L} - 3$$

vertices counted twice!

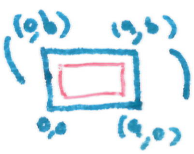
$L = \#\{(x,y) \in \mathbb{Z}^2_{\geq 0} : y - b = -\frac{b}{a}x\}$ $\qquad g = \gcd(b,a)$

$\#\{\ \underline{\qquad\qquad}\ : \frac{a}{g}(y-b) = -\frac{b}{g}x\}$

$\left(\frac{a}{g} : \frac{b}{g}\right) = 1 \Rightarrow y - b = k\left(\frac{-b}{g}\right)$ & $x = k\frac{a}{g}$ $\quad k \in \mathbb{Z}$ & $\boxed{0 \leq k \leq g}$

$\Rightarrow L = g + 1.$ & $\boxed{B(P) = a + b + g}$

$I(P) = ?$ $\rightsquigarrow$ $I(b\,\underset{a}{\square}) = 2I(P) + (\underset{\text{end pts of diag.}}{L - 2})$

$I($ 

$) = \begin{cases} 0 & \text{if } a = 1 \text{ or } b = 1 \\ (a-1)(b-1) & \text{if } a, b \geq 2 \end{cases}$

**Alternative**: $I(\square) = (a+1)(b+1) - B(\square) = (a+1)(b+1) - (2(a+1) + 2(b+1) - 4)$

$= a^2 + b^2 + a + b + 1 - 2a - 2b = (a-1)(b-1)$

So $\boxed{I(P) = \frac{(a-1)(b-1) - g + 1}{2}}$

$$\Rightarrow I(P) + \frac{B(P)}{2} - 1 = \frac{(a-1)(b-1) - g + 1}{2} + \frac{a + b + g}{2} - 1$$

$$= \frac{ab}{2} - \frac{a+b+g}{2} + 1 + \frac{a+b+g}{2} - 1 = \frac{ab}{2} = A(P) \qquad \square$$

**Obs**: Pick's Thm fails in dim 3. Reeve's Tetrahedron $= \operatorname{conv}\{(0,0,0), (1,0,0), (0,1,0), (1,1,h)\}$ for $h \in \mathbb{Z}_{>0}$ gives a counterexample $[\ \mathrm{Vol}(P) = h/6,\ I(P) = 0,\ B(P) = 4\ ]$ (HW6)

## §2 Ehrhart Theory in dimension 2:

Pick's Thm gives $L_P(1) = \#(P\cap\mathbb{Z}^2) = I(P)+B(P) = \boxed{A(P)+\frac{B(P)}{2}+1}$

**Theorem**: (1) $L_P(t) = \#(P\cap\frac{1}{t}\mathbb{Z}^2) = At^2+\frac{B}{2}t+1$ for $t\in\mathbb{Z}_{>0}$

(2) $L_{P^\circ}(t) = \#(P^\circ\cap\frac{1}{t}\mathbb{Z}^2) = L_P(-t)$

(3) $$\mathrm{Ehr}_P(z) = 1+\sum_{t\geq1}L_P(t)z^t = \frac{\overset{=h_2^*}{(A-\frac{B}{2}+1)}z^2+\overset{=h_1^*}{(A+\frac{B}{2}-2)}z+\overset{=h_0^*}{1}}{(1-z)^3}$$

(4) $h_1^* = L_P(1)-3\in\mathbb{Z}_{\geq0}$

$h_2^* = L_P(-1) = L_{P^\circ}(1)\in\mathbb{Z}_{\geq0}$.

**Proof**: (4) follows from (3) & formula for $L_P(1)$ & (2).

Fix $t\in\mathbb{Z}_{>0}$ (1). $\mathrm{Area}(tP) = t^2\,\mathrm{Area}(P)$ [show it for $\mathbb{Z}$-triangles in $\mathbb{R}^2$ + additivity of Areas by triangulations]

$B(tP) = |\partial(tP)\cap\mathbb{Z}^2| = B(P)\cdot t$ [show it for segments with vertices in $\mathbb{Z}^2$]

$(0,0)$ — $(a,b)$: $|\mathbb{Z}\text{-pts}|-1 = \gcd(a,b)$.

$\Rightarrow L_P(t) = I(tP)+B(tP) = A(tP)+\frac{B(tP)}{2}+1 = \boxed{t^2A+t\frac{B}{2}+1}$

Note: $L_P(0):=1$

Fix $t\in\mathbb{Z}_{>0}$ (2) $L_{P^\circ}(t) = L_P(t)-B(tP) = L_P(t)-tB = t^2A+t\frac{B}{2}+1-tB$
$= t^2A-\frac{B}{2}t+1 = L_P(-t)$.

(3) $$\mathrm{Ehr}_P(z) := \sum_{t\geq0}(At^2+\frac{B}{2}t+1)z^t$$
$$= A\sum_{t\geq0}t^2z^t+\frac{B}{2}\sum_{t\geq0}tz^t+\sum_{t\geq0}z^t \overset{(*)}{=} A\frac{z^2+z}{(1-z)^3}+\frac{B}{2}\frac{z}{(1-z)^2}+\frac{1}{(1-z)}$$
$$= \frac{(A-\frac{B}{2}+1)z^2+(A+\frac{B}{2}-2)z+1}{(1-z)^3}$$

(*) **Claim**: (1) $\sum_{t\geq0}t^2z^t = \frac{z^2+z}{(1-z)^3}$ & (2) $\sum_{t\geq0}tz^t = \frac{z}{(1-z)^2}$.

For (2): integrate & take derivative $\frac{1}{1-z} = \sum_{t\geq1}z^{t-1} \overset{?}{=} \int\frac{z}{(1-z)^2}dz + C$ & $\left(\frac{1}{1-z}\right)' = \frac{z}{(1-z)^2}$

For (1) Take derivative of (2): $\frac{1}{z}\sum_{t\geq0}t^2z^2 = \left(\frac{z}{(1-z)^2}\right)' = \frac{(1-z)^2+z\,2(1-z)}{(1-z)^4} = \frac{1-z+2z}{(1-z)^3} = \frac{1+z}{(1-z)^3}$

$\Rightarrow \sum_{t\geq0}t^2z^2 = \frac{z+z^2}{(1-z)^3}$.

∎

## §3 Integer-point Transforms for rational cones:

**Def** $S \subseteq \mathbb{R}^n$: $\sigma_S(z_1, \ldots, z_n) = \sum_{m \in S \cap \mathbb{Z}^n} \underline{z}^{\underline{m}}$ $\qquad \underline{z}^{\underline{m}} = z_1^{m_1} \cdots z_n^{m_n}$

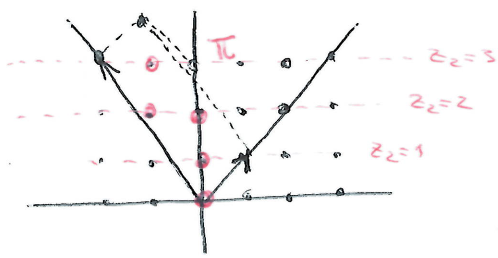
is the <u>integer-point transform</u> of <u>S</u> ( <u>moment</u> <u>generating</u> <u>function</u> of S) $\quad \underline{m} = (m_1, \ldots, m_n)$

<u>Example</u> 1: $K = [0, \infty) \subseteq \mathbb{R}$ 1-dim'l cone:

$$\sigma_K(z) = \sum_{m \geq 0} z^m = \frac{1}{1-z}$$

$$\sigma_{-\frac{1}{2}+K}(z) = \sum_{m \geq 0} z^m = \frac{1}{1-z}, \qquad \sigma_{\frac{1}{2}+K}(z) = \sum_{m \geq 1} z^m = \frac{z}{1-z}.$$

<u>Example 2</u>: $K = \mathbb{R}_{\geq 0}\langle (1,1), (-2,3) \rangle$




- $\Pi$ = fundamental parallelogram $= \{ \lambda_1(1,1) + \lambda_2(-2,3) : 0 \leq \lambda_1, \lambda_2 < 1 \}$
- $(\Pi + \mathbb{Z}_{\geq 0}\langle (1,1), (-2,3) \rangle)$ <u>tiles K</u> $\quad$ & $\mathbb{Z}^2 \cap \Pi$ has 5 points $= \{(0,0), (0,1), (0,2), (-1,2), (-1,3)\}$

$$\underset{\sigma_{\mathbb{Z}_{\geq 0}\langle (1,1),(-2,3) \rangle}}{\overset{\sigma_0(z)}{\shortparallel}} = \sum_{\substack{m = j(1,1) + \\ k(-2,3) \\ j,k \geq 0}} z^m = \sum_{j \geq 0} \sum_{k \geq 0} z^{j(1,1) + k(-2,3)} \underset{\text{geom series.}}{\downarrow=} \frac{1}{(1 - z_1 z_2)(1 - z_1^{-2} z_2^3)}$$

For each $(p,q) \in \Pi \cap \mathbb{Z}^2$, write $\mathcal{L}_{(p,q)} = \{ (p,q) + j(1,1) + k(-2,3) : j,k \in \mathbb{Z}_{\geq 0} \}$

<u>Claim</u>: $\bigsqcup_{(p,q) \in \Pi \cap \mathbb{Z}^2} \mathcal{L}_{(m,n)} = K \cap \mathbb{Z}^2$

$$\Rightarrow \sigma_K(z_1, z_2) = \big(\overbrace{1 + z_2 + z_2^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}^{\sigma_\Pi(z)}\big)\, \overset{\sigma_0}{\shortparallel}\!\!\sigma_{\mathcal{L}_{(0,0)}} = \frac{1 + z_2 + z_2^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}{(1 - z_1 z_2)(1 - z_1^{-2} z_2^3)}$$

<u>Key</u>: $f_{(z_2)} = \cancel{\sigma_K}(1, z_2) \in \mathbb{C}[[z_2]]$ & ~~with~~ $z_2^k$ in $f_{(z_2)}$ counts $(\mathbb{Z}^2 \cap K \cap (z_2 = k))$

- This extends to other <u>pointed</u> rational simplicial cones:

  ↳ Translate of a cone with apex 0 ( ie 0 is a face of the cone )

<u>Thm</u>: Fix $K = \mathbb{R}_{\geq 0}\langle w_1, \ldots, w_n \rangle$ n-dim'l cone with $w_1, \ldots, w_n \in \mathbb{Z}^n$ <u>primitive</u> ( gcd (words $w_i$) = 1 $\forall i$ ). For any $v \in \mathbb{R}^n$, we have:

$$\sigma_{v+K}(z) = \frac{\sigma_{v+\Pi}(z)}{(1 - z^{w_1}) \cdots (1 - z^{w_n})}$$

when $\Pi = \{ \sum \lambda_i w_i \mid 0 \leq \lambda_i < 1 \}$ is the <u>fundamental</u> <u>parallelepiped</u> of K

<u>Def</u>: K is <u>simplicial</u> ( $\dim(K) = |\text{Rays}(K)|$ )

Obs: If $\partial(v+K)\cap\mathbb{Z}^n=\emptyset$, we can replace $\Pi$ by $\Pi^\circ=\{\sum_{(*)}\lambda_i w_i \mid 0<\lambda_i<1 \ \forall i\}$

Proof of Thm: $\sigma_{v+K}(\underline{z})=\sum_{m\in(v+K)\cap\mathbb{Z}^n}\underline{z}^m$

Write $\underline{m}=v+\sum_{i=1}^n\lambda_i w_i \qquad \lambda_i\geq 0$

$K$ simplicial $\Rightarrow$ $\lambda_i$ are unique ( $\{w_1,\dots w_n\}$ are l.i.)

Write $\lambda_i=\underbrace{\lfloor\lambda_i\rfloor}_{\in\mathbb{Z}_{\geq 0}}+\underbrace{\{\lambda_i\}}_{\in[0,1)}$

$\Rightarrow m=\underbrace{\left(v+\sum_{i=1}^n\{\lambda_i\}w_i\right)}_{=:p\in v+\Pi}+\underbrace{\sum_{i=1}^n\lfloor\lambda_i\rfloor w_i}_{\in\mathbb{Z}_{\geq 0}\langle w_1,\dots w_n\rangle}$

Note: $m,\ m-p\in\mathbb{Z}^n \Rightarrow p\in(v+\Pi)\cap\mathbb{Z}^n$ & is uniquely assoc to $\underline{m}$.

Conclusion: We've defined a bijection

$$\varphi:(v+K)\cap\mathbb{Z}^n \longrightarrow (v+\Pi)\cap\mathbb{Z}^n\times\mathbb{Z}_{\geq 0}\langle w_1,\dots w_n\rangle$$
$$\underline{m}\longmapsto(\underline{p},\ \underline{m}-\underline{p})$$

So $\sigma_{v+K}(z)=\sigma_{v+\Pi}(z)\cdot\sum_{\underline{\lambda}\in\mathbb{Z}^n_{\geq 0}}\underline{z}^{\sum_{i=1}^n\lambda_i w_i}$

$$=\sigma_{v+\Pi}(z)\,\frac{1}{\prod_{i=1}^n(1-\underline{z}^{w_i})}$$

germ series

$\sigma_{v+\Pi}(z)\in\mathbb{Z}_{\geq 0}[z_1^{\pm},\dots,z_n^{\pm}]$ ( $(v+\Pi)\cap\mathbb{Z}^n$ is finite!) $\blacksquare$

Next Time: Use this to prove Ehrhart Theorem for lattice simplicial polytopes..