Lecture XXIX: Ehrhart Theory for simplicial polytopes.

Recall: $K = \mathbb{R}_{\geq 0}\langle w_1, \ldots, w_N\rangle$ N-dim'l pointed simplicial cone in $\mathbb{R}^N$ with $w_1, \ldots, w_N \in \mathbb{Z}^N$ (primitive) (rat'l simplicial, with apex $= \underline{0}$)

Thm 1: For any $v \in \mathbb{R}^n$
$$\sigma_{v+K}(\underline{z}) = \frac{\boxed{\sigma_{v+\Pi}(\underline{z})}}{(1-\underline{z}^{w_1})\cdots(1-\underline{z}^{w_N})} \qquad \underline{z} = (z_1, \ldots, z_N)$$
$= \sum_{m \in (v+K)\cap\mathbb{Z}^N} \underline{z}^{\underline{m}}$; $\sigma_{v+\Pi}(\underline{z}) \in \mathbb{Z}_{\geq 0}[z_1^{\pm}, \ldots, z_n^{\pm}]$ because $v+\Pi$ is bounded

where $\Pi = \{\sum \lambda_i w_i \mid 0 \leq \lambda_i < 1\}$ is the fundamental parallelepiped of K.
(Π really depends on $w_1 \ldots w_n$)

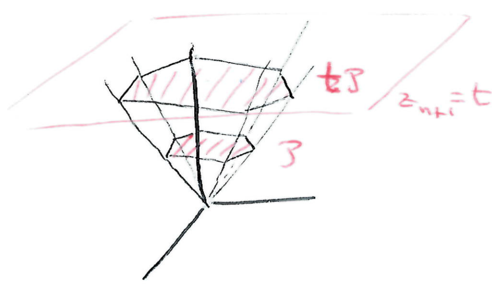
Obs: If $\partial(v+K) \cap \mathbb{Z}^N = \emptyset \Rightarrow$ can replace Π by $\Pi^\circ = \{\sum \lambda_i w_i \mid 0 < \lambda_i < 1\}$

Note if $w_1, \ldots, w_N$ are not primitive, we get a sub-optimal description b/c cancellations in (RHS) will occur.

§1 Application: Ehrhart Series for simplicial lattice polytopes

Fix $\mathcal{P} \subseteq \mathbb{R}^n$ d-dim'l polytope with $V(\mathcal{P}) = \{v_1, \ldots, v_m\}$ $(m \geq d+1)$

Def $\operatorname{Cone}(\mathcal{P}) = \mathbb{R}_{\geq 0}\langle (v_1, 1), \ldots, (v_m, 1)\rangle \subseteq \mathbb{R}^{n+1}$



Note (1) $\dim(\mathcal{P}) = d \Rightarrow \dim(\operatorname{cone}(\mathcal{P})) = d+1$

(2) $\operatorname{Rays}(\mathcal{P}) = \{(v, 1) : v \in V(\mathcal{P})\}$

(3) $\mathcal{P} \longleftrightarrow \operatorname{cone}(\mathcal{P}) \cap (z_{n+1} = 1)$

In general $\operatorname{Cone}(\mathcal{P}) \cap \{z_{n+1} = t\} = t\mathcal{P} \times \{t\}$ $\forall t \in \mathbb{R}_{>0}$.

(4): If $\mathcal{P}$ is a lattice polytope $\Rightarrow \operatorname{Cone}(\mathcal{P})$ is a rational polytope with apex $\underline{0}$

If $\mathcal{P}$ is simplicial $(m = d+1) \Rightarrow \operatorname{Cone}(\mathcal{P})$ is simplicial.

Consequence: We can use $\sigma_{\operatorname{Cone}(\mathcal{P})}(z_1 \ldots z_{n+1})$ to count points in $\mathbb{Z}^{n+1} \cap \operatorname{cone}(\mathcal{P})$.

Prop: Assume $\mathcal{P}$ is any lattice full-dim'l polytope in $\mathbb{Z}^n$.

- $\sigma_{\operatorname{cone}(\mathcal{P})}(z_1, \ldots, z_{n+1}) = \sum_{m \in \operatorname{Cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}} \underline{z}^{\underline{m}} = 1 + \sum_{t \geq 1} \underbrace{\Big(\sum_{\underline{m}' \in t\mathcal{P} \cap \mathbb{Z}^n} \underline{z}^{\underline{m}'}\Big)}_{= \sigma_{t\mathcal{P}}(z_1, \ldots, z_n)} z_{n+1}^t$ (values for $m_{n+1} \in \mathbb{Z}_{\geq 0}$)
- $\sigma_{t\mathcal{P}}(\underline{1}) = L_{\mathcal{P}}(t) \Rightarrow \boxed{\sigma_{\operatorname{Cone}(\mathcal{P})}(1, \ldots, 1, z) = \operatorname{Ehr}_{\mathcal{P}}(z)}$

Next: assume $\mathcal{P}$ is simplicial ($|V(\mathcal{P})| = n+1$).

Using Thm, writing $\operatorname{cone}(\mathcal{P}) = \mathbb{R}_{\geq 0}\langle \underbrace{(v_1, 1)}_{= w_1}, \ldots, \underbrace{(v_{n+1}, 1)}_{w_{n+1}}\rangle$ $w_i \in \mathbb{Z}^{n+1}$ primitive.

We get
$$\boxed{\sigma_{\operatorname{cone}(\mathcal{P})}(\underline{z}, z_{n+1}) = \frac{\sigma_{\Pi}(\underline{z}, z_{n+1})}{(1 - \underline{z}^{v_1} z_{n+1}) \cdots (1 - \underline{z}^{v_{n+1}} z_{n+1})}} \qquad (*)$$

Thm 2: $\mathcal{P}$ lattice full-dim'l simplicial polytope in $\mathbb{R}^n$. Then:

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sigma_{\Pi}(1,\ldots,1,z)}{(1-z)^{n+1}}$$ where $\sigma_{\Pi}(1,\ldots,1,z) \in \mathbb{Z}_{\geq 0}[z]$ has degree $\leq n$ in z. & $\sigma_{\Pi}(1,\ldots,1,1) \neq 0$

($\therefore$ Ehrhart Series Thm + Non-negativity of $h^*_{\mathcal{P}}$)

Proof: By $(*)$, we can specialize (RHS) at $\underline{z} = (1,\ldots,1)$ & $z_{n+1} = z$.

. Denominator becomes $(1-z)^{n+1}$.

. Numerator: $\sigma_{\Pi}(\underline{z}, z_{n+1}) \in \mathbb{Z}_{\geq 0}[z_1^{\pm},\ldots,z_n^{\pm}, z_{n+1}]$ by construction

For $p \in \Pi \cap \mathbb{Z}^{n+1}$: $p = (p_1,\ldots,p_{n+1}) = \sum_{i=1}^{n+1} \lambda_i (v_i, 1)$ $\quad 0 \leq \lambda_i < 1$

$\Rightarrow p_{n+1} = \sum_{i=1}^{n+1} \lambda_i < n+1$ & in $\mathbb{Z}$ $\Rightarrow p_{n+1} \leq n$.

So all monomials in $\sigma_{\Pi}(\underline{z}, z_{n+1})$ have z_{n+1} degree $\leq n$.

Evaluation at $(1,\ldots,1,z)$ gives $\sigma_{\Pi}(1,\ldots,1,z) \in \mathbb{Z}_{\geq 0}[z]$ of deg $\leq n$. ∎

Corollary: $\mathcal{P}$ lattice, full dim'l simplicial polytope in $\mathbb{R}^n$, then:

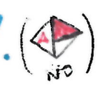
$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{j=0}^{n} h_j z^j}{(1-z)^{n+1}} \quad \& \quad h_k = \#\Big(\mathbb{Z}^{n+1} \cap \Big\{\sum_{i=1}^{n+1} \lambda_i (v_i, 1) \;\Big|\; 0 \leq \lambda_i < 1\Big\} \cap \{z_{n+1} = k\}\Big)$$

§2 Ehrhart Series & Stanley's non-negativity for lattice polytopes:

We exploit the result for simplicial polytopes by triangulations.

Def: A triangulation of a d-dim'l polytope $\mathcal{P}$ in $\mathbb{R}^n$ is a finite collection $\mathcal{T}$ of d-simplices satisfying:

(1) $\mathcal{P} = \bigcup_{\Delta \in \mathcal{T}} \Delta$

(2) For $\Delta, \Delta' \in \mathcal{T}$: $\Delta \cap \Delta'$ is a common face of both Δ & Δ'. ()

If $V(\Delta) \subseteq V(\mathcal{P})$ $\forall \Delta \in \mathcal{T}$, we say the triangulation $\mathcal{T}$ uses no new vertices

Obs: Coning over $\mathcal{P}$ & each Δ gives the definition of a triangulation for a pointed cone with apex $\underline{0}$ ("no new vertices" condition becomes "no new rays")

Thm 3: Every polytope (resp. pointed cone) in $\mathbb{R}^n$ can be triangulated using no new vertices (resp rays)

Proof: Up to translation & $SL(n,\mathbb{R})$-action, every pointed cone with apex $\underline{0}$ is $\operatorname{cone}(\mathcal{P})$ for some polytope $\mathcal{P}$, so it's enough to prove ~~this~~ for polytopes.

We set $d=\dim(\mathcal{P})$ & induct on $|V(\mathcal{P})|\geq d+1$.

- If $|V(\mathcal{P})|=d+1$, then $\mathcal{P}$ is a simplex so $\mathcal{T}=\{\mathcal{P}\}$.
- If $|V(\mathcal{P})|>d+1$, assume the statement holds for all polytopes Q with $\dim Q=d$ & $|V(Q)|<|V(\mathcal{P})|$.

Since $|V(\mathcal{P})|>\dim\mathcal{P}+1$, $\exists\, v\in V(\mathcal{P})$ s.t. $Q=\operatorname{conv}((V(\mathcal{P})\smallsetminus\{v\}))$ has $\dim Q=d$. <u>Note</u> $V(Q)\subseteq V(\mathcal{P})\smallsetminus\{v\}$.

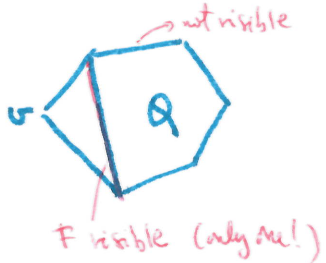
By our (IH), we can triangulate Q with no new vertices by a collection $\mathcal{T}$.

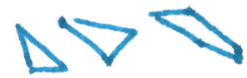
We want to use this to triangulate $\mathcal{P}$. To this end, we ~~define~~ must pick suitable <u>facets</u> of Q.

Def A facet F of Q (ie $(\dim Q-1)$-face of Q) is <u>visible</u> from v if $\forall x\in F$: $[x,v]\cap Q=\{x\}$

↳ segment joining x & v in $\mathbb{R}^n$.

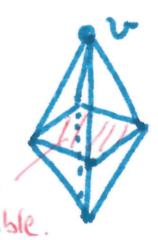
Eg: ①



$\mathcal{T}=$ 

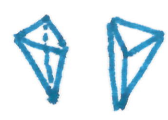
$\mathcal{T}_F=\{F\}$ ⇝ $v * F$ missing simplex to triangulate $\mathcal{P}$

②



$\mathcal{T}_F$:  ⇝ 

→ missing simplices to triangulate $\mathcal{P}$

$\mathcal{T}=$ 

Claim 1 If F is a facet of Q: $\mathcal{T}_F=\{\Delta_F:=\Delta\cap F \mid \Delta\in\mathcal{T},\ \dim\Delta_F=d-1\}$ is a triangulation of F using only vertices of F.

Claim 2: $\mathcal{T}^*=\mathcal{T}\cup\bigcup_{F\text{ visible from }v}\{\operatorname{conv}(\Delta_F,v):\Delta_F\in\mathcal{T}_F\}$ is a triangulation of $\mathcal{P}$ with no new vertices.

[For details, see Appendix B of Beck, Sinai (2007)]