

# Lecture XXX:

Last time: defined triangulation of polytopes (pointed cones) using no new vertices, (resp. rays)

- $\text{unk}(\mathcal{P}) = \{ (v_1, 1), \dots, (v_m, 1) \} \subseteq \mathbb{R}^{n+1}$  for  $\mathcal{P} \subseteq \mathbb{R}^n$  polytope with  $v(\mathcal{P}) = \{v_1, \dots, v_m\}$
- $\sigma_{\text{cone}(\mathcal{P})}(1, \dots, 1, z) = \text{Ehr}_{\mathcal{P}}(z)$

Ehrhart series, Ehrhart polynomial & Stanley's non-negativity holds for simplicial lattice polytopes in  $\mathbb{R}^n$


## §1. Ehrhart Theory for lattice polytopes

Fix  $\mathcal{P} \subseteq \mathbb{R}^n$   $\dim \mathcal{P} = d$  lattice polytope

1. We will prove the statement for Ehrhart polynomials:  $L_{\mathcal{P}}(t) \in \mathbb{Q}[t]$  of  $\deg = d$ .
2. Show Ehr polynomial  $\Rightarrow$  Ehr series Thm
3. Prove Stanley's non-negativity.

Thm 1:  $L_{\mathcal{P}}(t) \in \mathbb{Q}[t]$  of  $\deg = d$ .

Proof: Know this holds for  $\mathcal{P}$  simplicial:  $\text{Ehr}_{\mathcal{P}}(z) = \sum_{j=0}^d \frac{h_j^* z^j}{(1-z)^{d+1}} \Rightarrow L_{\mathcal{P}}(t) = \binom{t+d}{d} + \sum_{j=1}^d h_j^* \binom{t+h_j}{n}$   
 leading coeff =  $\text{vol}(\mathcal{P})$   
 $\in \mathbb{Q}[t]$  of  $\deg = d$ .

(1)  $\dim \mathcal{P} = 1$ , we translate by a point in  $\mathbb{Z}^n$  & Take the linear span rather than  $\mathbb{R}^n$   
 To reduce to   $L_{\mathcal{P}}(t) = |\mathcal{P} \cap \frac{1}{t}\mathbb{Z}| = |t\mathcal{P} \cap \mathbb{Z}| = ta + 1$  ✓

(2) Assume  $\dim \mathcal{P} > 1$  &  $\mathcal{P}$  is not simplicial.

We fix  $\mathcal{T} = \{\Delta_1, \dots, \Delta_m\}$  a triangulation of  $\mathcal{P}$  with no new vertices.

By Inclusion - Exclusion:  $\mathcal{P} = \bigcup_{i=1}^m \Delta_i$

$$L_{\mathcal{P}}(t) = \sum_{i=1}^m L_{\Delta_i}(t) + \sum_{s=2}^m (-1)^{s-1} \sum_{\substack{i_1, \dots, i_s \\ \Delta_{i_1} \cap \dots \cap \Delta_{i_s} \neq \emptyset}} L_{\Delta_{i_1} \cap \dots \cap \Delta_{i_s}}(t)$$

- $\Delta_i$  d-simplex  $\Rightarrow L_{\Delta_i}(t) \in \mathbb{Q}[t]$  of degree d & leading coeff =  $\frac{1}{d!} = \text{vol}(\Delta_i)$
- $\Delta_{i_1} \cap \dots \cap \Delta_{i_s}$  are also simplices (face of a simplex is a simplex)

$$\Rightarrow L_{\Delta_{i_1} \cap \dots \cap \Delta_{i_s}}(t) \in \mathbb{Q}[t] \text{ of } \deg = l = \dim \Delta_{i_1} \cap \dots \cap \Delta_{i_s} < d$$

To go from the Ehrhart polynomial to the Ehrhart series, we need the following lemma

Lemma: If  $\sum_{t \geq 0} f(t) z^t = \frac{g(z)}{(1-z)^{d+1}}$ , then " $f \in \mathbb{Q}[t]$  of  $\deg f \leq d$ "

Moreover if  $f \in \mathbb{Q}[t]$  has  $\deg f = d$ ,  $\sum_{t \geq 0} f(t) z^t \stackrel{\text{LHS description}}{\iff} g \in \mathbb{Q}[z]$  of  $\deg \leq d$  &  $g(1) \neq 0$ .

3f/ ( $\Leftarrow$ ) Was done in Lecture 27. If  $g(z) = \sum_{i=0}^d h_i z^i$  then:

$$f(t) = \sum_{j=0}^d h_j \binom{t+n-j}{n} \in \mathbb{Q}[t] \quad \binom{t+n-j}{n} = \binom{t-j+n}{n} = \frac{(t-j+n) \cdots (t-j+1)}{n!}$$

But  $\text{coeff}(f(t)) = \sum_{j=0}^d h_j \frac{1}{n!} = \frac{g(1)}{n!} \neq 0 \Rightarrow \deg f = n$

Alternative:  $f(t)$  satisfies the linear recursion given by  $(1-z)^{d+1} = 1 + \sum_{i=1}^d a_i z^i$  &  $f(t) = p(t) z^t$   $\deg p < d+1$

( $\Rightarrow$ ) Write  $f(t) = \sum_{j=0}^d a_j t^j$  with  $a_d \neq 0$ .

[+ known statement]

Then  $\sum_{t \geq 0} f(t) z^t = \sum_{m=0}^d a_m \left( \sum_{t \geq 0} t^m z^t \right)$

[By HW6 Problem]  $\sum_{t \geq 0} t^m z^t = \sum_{k=0}^m \boxed{A(m, k)} z^k$  with  $A(m, k) \in \mathbb{Z}_{\geq 0}$  for  $0 \leq k \leq m$ .   
  $A(m, m) = 1 \quad \forall m \geq 0$ .   
 *Eulerian numbers*

$$\text{So } \sum_{t \geq 0} f(t) z^t = \sum_{m=0}^d a_m \sum_{k=0}^m A(m, k) z^k \frac{(1-z)^{m+1}}{(1-z)^{m+1}} = \frac{\sum_{m=0}^d a_m \sum_{k=0}^m A(m, k) z^k (1-z)^{d-m}}{(1-z)^{d+1}}$$

So  $g(z) = \sum_{m=0}^d a_m \sum_{k=0}^m A(m, k) z^k (1-z)^{d-m}$  has degree  $\leq d$ .

$$g(1) = a_d \sum_{\substack{k=0 \\ > 0}}^d A(d, k) \neq 0$$

Thm 2:  $\text{Ehr}_g(z) = \frac{\sum_{k=0}^d h_k^*(z) z^k}{(1-z)^{d+1}}$

3f/ The last lemma (+ Thm 1) w/  $f = L_g(t) \in \mathbb{Q}[t]$ ,  $\deg f = d$  gives

$\text{Ehr}_g(z) = \sum_{t \geq 0} f(t) z^t = \frac{g(t)}{(1-z)^{d+1}}$  with  $g(t) \in \mathbb{Q}$  of  $\deg \leq d$  &  $g(1) \neq 0$ .

$\Rightarrow g(t) = \text{numerator in the (RHS)}$ . Only missing thing: argue that  $L_g(0) = 1$ . This follows from proof of Thm 3 ( $h_0^*(z) = 1$ ).

Thm 3 (Stanley's  $m$ -negativity)  $h_0^*(z), \dots, h_d^*(z) \in \mathbb{Z}_{\geq 0}$  (ie coeffs of numerator of  $\sum_{t \geq 0} L_g(t) z^t$ )

Proof: Set  $K = \text{cone}(B) \subseteq \mathbb{R}^{n+1}$  & triangulate  $K$  into rat'l simplicial cones with apex 0 using no new rays  $\gamma = \{K_1, \dots, K_m\}$

Claim:  $\exists w \in \mathbb{R}^{n+1}$  with  $w_{n+1} < 0$  &  $w_i \neq 0 \quad \forall i$  st  
 (1)  $\text{cone}(B) \cap \mathbb{Z}^{n+1} = (w + \text{cone}(B)) \cap \mathbb{Z}^{n+1}$  & (2)  $\partial(w + K_i) \cap \mathbb{Z}^{n+1} = \emptyset \quad \forall i = 1, \dots, m$ .



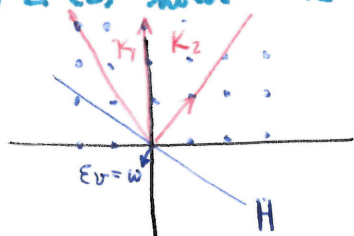
3f/  $-e_{n+1}$  defines a supporting hyperplane for  $\emptyset$  ( $H(x_{n+1}=0)$ ). Pick  $v = \max -e_{n+1}$  in  $\mathbb{Q}^n$  &  $0 < \epsilon < 1$  small enough with  $\epsilon \notin \mathbb{Q}$ .

Then  $w = \epsilon v$  satisfies  $H = \langle v \rangle^\perp$  supports  $\emptyset$

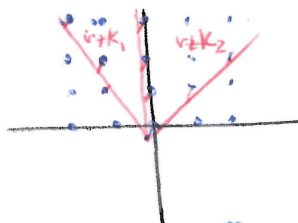
$\bullet$   $H$  separates  $K$  from  $w$

(1) & (2) hold b/c all words of  $w$  are irrational but  $K_i$  are rat'l &  $\epsilon$  is small enough.

Ex:



$\rightsquigarrow$



(Small enough  $\epsilon$  exists because  $d(\partial K_i, \mathbb{Z}^n, \mathbb{Z}^n) > 0 \quad \forall i$ )

$\hookrightarrow$  to not get new points in  $(w + \text{cone}(\mathcal{B})) \cap \mathbb{Z}^{n+1}$ .

$\bullet$  To get no lattice pts in  $\partial(w + K_i)$   $\square$

Claim has the following consequence:

$$p \in (w + \text{cone}(\mathcal{B})) \cap \mathbb{Z}^{n+1} \iff \exists ! i=1, \dots, m \text{ st } p \in (w + K_i) \cap \mathbb{Z}^{n+1}$$

$$\text{So } (w + \text{cone}(\mathcal{B})) \cap \mathbb{Z}^{n+1} = \bigsqcup_{i=1}^m (w + K_i) \cap \mathbb{Z}^{n+1}$$

$$\text{& } \sigma_{\text{cone}(\mathcal{B})}(z) = \sigma_{w + \text{cone}(\mathcal{B})}(z) = \sum_{i=1}^m \sigma_{w + K_i}(z) \quad (\text{no PIE needed!})$$

$$\text{So Ehr}_\mathcal{B}(z) = \sigma_{\text{cone}(\mathcal{B})}(1, \dots, 1, z) = \sum_{i=1}^m \sigma_{w + K_i}(1, \dots, 1, z)$$

$$\text{To finish, use } K_i \subseteq \mathbb{R}_{\geq 0}^{d+1} \cdot (v_j^{(i)}, 1) \Rightarrow \sigma_{w + K_i}(1, \dots, 1, z) = \frac{\sigma_{w + \pi_i}(1, \dots, 1, z)}{(1-z)^{d+1}} \quad (\text{Lecture 29})$$

$\text{for } v_1^{(i)}, \dots, v_{d+1}^{(i)} \in V(\mathcal{B})$

$\hookrightarrow \pi_i = \text{multiplicities of } K_i$

Since  $\partial(w + K_i) \cap \mathbb{Z}^{n+1}$ , can replace  $\pi_i$  by  $\pi_i^0 = \{ \sum_{j=1}^{d+1} \lambda_j (v_j^{(i)}, 1) \mid 0 < \lambda_j < 1 \}$

Now:  $\sigma_{w + \pi_i^0}(1, \dots, 1, z) \in \mathbb{Z}_{\geq 0}[z]$  by construction.

$$\text{Since Ehr}_\mathcal{B}(z) = \sum_{i=1}^m \sigma_{w + \pi_i^0}(1, \dots, 1, z) \in \mathbb{Z}_{\geq 0}[z]$$

, then  $\text{numerator} \in \mathbb{Z}_{\geq 0}[z]$  is the one that we wanted (by Thm 2).

Apex of  $K$  in any cone  $w + \pi_i^0$   
§2 From Lattice To Rational polytopes  $\xrightarrow{(\text{any one})}$   $\sigma_{w + \pi_i^0}(1, \dots, 1, z)$  has constant term  $\neq 0$  ( $j=i$ ) & it's value is  $1 = h_0^*(\mathcal{B})$ .

Fix  $\mathcal{B} \in \mathbb{R}^n$  with  $V(\mathcal{B}) \subseteq \mathbb{Q}^n$   $\dim(\mathcal{B}) = d \Rightarrow$  define  $L_\mathcal{B}(t), L_{\mathcal{B}^0}(t), \text{Ehr}_\mathcal{B}(z)$

Def  $D = \text{Denominator of } \mathcal{B} = \min \{ k \in \mathbb{Z}_{>0} \mid k\mathcal{B} \text{ is a lattice polytope} \} = \text{lcm}(\text{denoms of } v \in V(\mathcal{B}))$

Def: A quasipolynomial  $Q$  is an expression of the form,

$$Q(t) = c_n(t) t^n + \dots + c_1(t) t + c_0(t) \quad t \in \mathbb{Z}_{\geq 0}$$

where  $c_i: \mathbb{Z} \rightarrow \mathbb{Q}$  are periodic functions of  $t$  and  $c_n(t) \neq 0$ .

$K = \text{Period of } Q = \text{least common period of } c_1, \dots, c_n$

Equivalently:  $\exists K$  many polynomials  $p_0(t), \dots, p_{K-1}(t) \in \mathbb{Q}[t]$  with  $Q(t) = p_i(t)$  if  $t \equiv i \pmod K$

4 main results:

Thm 1 (Ehrhart quasipolynomials): If  $P$  is a rational polytope in  $\mathbb{R}^n$ , then  $L_P(t)$  is a quasipolynomial in  $t$  of degree  $d$ . Its period divides the denominator of  $P$ . Furthermore,  $c_d(t) = \text{Vol}(P)$  is constant.

Thm 2 (Ehrhart-Macdonald Reciprocity)  $L_{P^\circ}(t) = (-1)^{\dim P} L_P(t)$   $\forall t \in \mathbb{Z}_{\geq 0}$

Thm 3 (Ehrhart Series)  $\text{Ehr}_P(z) = \frac{g(z)}{(1-z^D)^{d+1}}$  for some  $g(z) \in \mathbb{Q}[z]$  of degree  $g < D(d+1)$  where  $D = \text{denominator}(P)$ . ( $h^*$ -polynomial of  $P$ )

Thm 4 (Stanley's non-negativity):  $g(z) = \sum_{j=0}^{D(d+1)} h_j^*(P) z^j$  with  $h_j^*(P) \in \mathbb{Z}_{\geq 0} \forall j$

Key steps: ① Prove it for simplicial rational polytopes:  $\Delta$   $V(\Delta) = \{v_1, \dots, v_{d+1}\}$  (Thm 3.2.4)

$K = \text{cone}(\Delta) = \mathbb{R}_{\geq 0} \langle (v_1, 1), \dots, (v_{d+1}, 1) \rangle = \mathbb{R}_{\geq 0} \langle (Dv_1, D), \dots, (Dv_{d+1}, D) \rangle$  with  $w_i \in \mathbb{Z}^n$ . Take  $\Pi =$  fund. parallelepiped for  $w_1, \dots, w_{d+1}$ .

$$\Rightarrow \sigma_K(z_1, \dots, z_{d+1}) = \frac{\sigma_\Pi(z_1, \dots, z_{d+1})}{(1-z^{w_1}) \dots (1-z^{w_{d+1}})}$$

$$m = (m_1, \dots, m_{d+1}) \in \Pi \cap \mathbb{Z}^{n+1} \quad m = \sum_{i=1}^{d+1} \lambda_i (w_i, D) \Rightarrow m_{d+1} = D \sum_{i=1}^{d+1} \lambda_i \leq D(d+1)$$

$\Rightarrow \sigma_\Pi(1, \dots, 1, z) \in \mathbb{Z}_{\geq 0}[z]$  of degree  $\leq D(d+1)-1$

$$\text{Ehr}_\Delta(z) = \sigma_K(1, \dots, 1, z) = \frac{\sigma_\Pi(1, \dots, 1, z)}{(1-z^D)^{d+1}} = g(z) \text{ in the statement.}$$

② Use PIE to conclude Thm 1 for simplices  $\Rightarrow$  Thm 1 in general.

③ Need a lemma bridging Thm 1 & Thm 3. Triangulate  $P$  to conclude Thm 3 from the simplicial case of Thm 3 + Thm 1 in general.

L30/5

Lemma: Assume  $\sum_{t \geq 0} f(t) z^t = \frac{g(z)}{h(z)}$  is a rational function (in reduced form)

and  $f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$ . TFAE:

(1)  $f$  is a quasipolynomial of degree  $d$  with period dividing  $P$ .

(2)  $g$  &  $h$  are polynomials such that

- $\deg(g) < \deg(h)$
- All roots of  $h$  are  $P^{\text{th}}$  roots of unity with multiplicity  $\leq d+1$
- $\exists$   $P^{\text{th}}$  root of 1 that has mult  $= d+1$  as a root of  $h$ .

④ Use perturbation argument to deduce Thm 4 for rat'l polytopes from the simplicial case.