Lecture XXXI: Basics on posets, induced structures, new posets from old

Why study posets?

① Many combinatorial objects admit meaningful poset structures

② Möbius inversion for posets generalizes:

(1) finite difference operators

(2) PIE

(3) Möbius function in Number Theory

③ Deep connections with algebra & topology (eg Hyperplane arrangements), applicable to various classes of generating functions

§1 Basic definitions & examples

Definition: A partially ordered set (poset) is a set $\mathcal{P}$ together with a binary operation $\leq$ (or $\leq_{\mathcal{P}}$) satisfying:

(1) [Reflexivity] For all $t \in \mathcal{P}$, $t \leq t$

(2) [Antisymmetry] If $s \leq t$ & $t \leq s$, then $t = s$

(3) [Transitivity] If $s \leq t$ & $t \leq u$, then $s \leq u$.

Obs: · $s \leq t$ & $s \neq t$, write $s < t$

· If $s \leq t$ or $t \leq s$, we say s & t are comparable. If s & t are not related

↑ accent this syllable

they are incomparable (notation: $s \| t$)

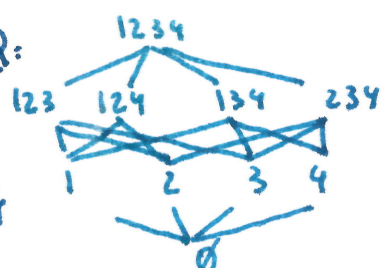
↑ accent this syllable.

Def: Order $\leq$ is total if $\forall s, t \in \mathcal{P}$: $s \leq t$ or $t \leq s$. or $s = t$. (eg: chains) (TOTAL)

Examples: ① $[n] = \{1, \dots, n\}$ with usual $\leq$. [n chain]

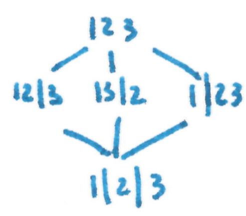
② [Boolean algebra] $\mathcal{B}_n = 2^{[n]}$ with $S \leq T$ if $S \subseteq T$ (inclusion relation)

$B_4 \supseteq Q$:

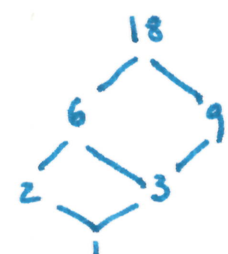


$Q = \{S \subseteq B_4 \mid |S| \neq 2\}$ induced $\leq$.

Π_3:



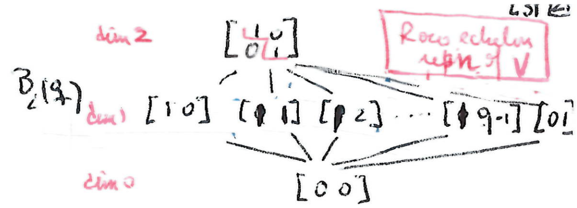
D_{18}:



③ D_n = pos. divisors of n, with the "divisibility order" $i \leq j$ if $i \mid j$ ("i divides j") [Divisor Lattice]

④ Π_n = Partition ~~lattice~~ = partitions of $[n]$ where $\pi \leq \sigma$ if every block of π is in a block of σ ("refinement order") Eg $\{\{1\}, \{2,3,4\}, \{5,6\}\} \leq \{\{1\}, \{2\text{-}6\}\}$

5. Any collection of sets ordered by inclusion



Es $\mathcal{B}_n(q) = \{ V \subseteq \mathbb{F}_q^n$ vector subspace $\}$
[q-analog of B_n]

Q: How to draw posets? Use Haase diagrams!

Def: For $s, t \in P$, we say <u>t covers s</u> if $s < t$ and there is no element $z \in P$ with $s < z < t$. Write $s \lessdot t$ or $s \prec t$ ("\prec" in LaTeX)

Def: The <u>Haase diagram</u> of a finite poset P is a directed graph H with

- $V(H)$ = elements of P
- $E(H) = (s,t)$ if $s \lessdot t$ [draw s below t to indicate the direction of the arrows]

[idea: minimal edges to get all relations by transitivity]

Es:

$|P| = 4$: 16 total. YES , NO , YES , YES , other examples from before

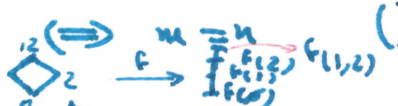
2 Algebraic structures

Def: Two posets P & Q are <u>isomorphic</u> (write $P \simeq Q$) if there is an order-preserving bijection $\phi : P \longrightarrow Q$ whose inverse is also order-preserving:
($s \leq_P t \Rightarrow \phi(s) \leq_Q \phi(t)$)

$$s \leq t \text{ in } P \iff \phi(s) \leq \phi(t) \text{ in } Q$$

Eg: $B_m \simeq B_n \iff m = n$ (In general: $B_S = 2^S \simeq B_T \iff |S| = |T|$)

Obs: ϕ $B_2 \longrightarrow [4]$



(order preserving map, NOT iso)

Def. A <u>weak subposet</u> Q of P is a subset Q of P with $\leq_Q$ satisfying

$$s \leq t \text{ in } Q \implies s \leq t \text{ in } P$$

(Things related in P may not be related in Q)

- A <u>refinement</u> of P is a poset P s.t. Q is weak subposet of P with $Q = P$ as sets.
- An <u>(induced) subposet</u> Q of P is a subset Q of P with the relation
  $s, t \in Q$ and $s \leq t$ in $P \Rightarrow s \leq t$ in Q. (order relation descends to Q)

<u>Example</u>: x, y in P ⤳ <u>closed intervals</u> from x to y

$[x,y] = \{ z \in P : x \leq z \leq y \}$ (closed)

$(x,y) = \{ z \in P : x < z < y \}$ (open)

induced subposets of P.

Def $Q \subseteq P$ is a <u>convex subposet</u> of P if $x < y$ in $Q \Rightarrow (x,y)_P \subseteq Q$.

Def: A poset P is <u>locally finite</u> if every interval of P is finite.

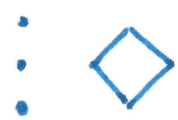
§3. New posets from old.

Fix P, Q 2 posets

① <u>Disjoint union</u> $P+Q = (\underset{\text{(disj union)}}{P \sqcup Q}, \leq)$ with

$$x \leq y \text{ iff } \begin{cases} x, y \in P \ \& \ x \leq_P y \\ x, y \in Q \ \& \ x \leq_Q y \end{cases}$$

<u>Notation</u>: $nP = \underbrace{P + \cdots + P}_{n \text{ times}}$

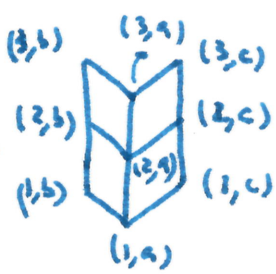
<u>Ex</u>: $3 + B_2 =$  (disj union of Hasse diagrams)

② <u>Cartesian Product</u> $P \times Q = (P \times Q, \leq)$ with

$(p,q) \leq (p',q')$ iff $p \leq_P p'$ & $q \leq_Q q'$.

<u>Notation</u> $P^n = \underbrace{P \times \cdots \times P}_{n \text{ times}}$

<u>Ex</u>: $3 \times$ 

$=$ 

- Draw H_P & replace each vertex p by a copy Q_p of Hasse diag. of Q
- Connect copies Q_p & Q_t (point by point) if $p \lessdot t$ in P.

<u>Obs</u>: $P \times Q \simeq Q \times P$ but Hasse diagram can look very different, even though they must be isomorphic

③ <u>Ordinal Sum</u>: $P \oplus Q = (P \sqcup Q, \leq)$ with

$$x \leq y \text{ if } \begin{cases} x, y \in P \ \& \ x \leq_P y \\ x, y \in Q \ \& \ x \leq_Q y \\ x \in P \ \& \ y \in Q \end{cases}$$

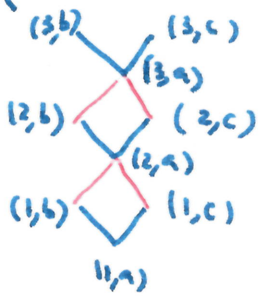
<u>Ex</u>: $n\text{-chain} = \underbrace{1 \oplus 1 \oplus \cdots \oplus 1}_{n \text{ copies}}$

④ Ordinal Product

$P \otimes Q = (P \times Q, \le)$ with (= coordinatewise &)

$$(p,q) \le (p',q') \text{ if } \begin{cases} p = p' \ \& \ q < q' \\ p < p' \end{cases}$$

Ex $3 \otimes$ 

 = 

- Draw H_P & replace each vertex p by a copy Q_p of H_Q.
- Connect minimal elements in Q_s to maximal elements in Q_t if $t \lessdot s$.

⑤ Power Posets

P^Q with

(1) Elements = order-preserving maps $f : Q \longrightarrow P$

(2) $f \le g$ iff $f(x) \le g(x)$ $\forall x \in Q$

Ex: $P =$ 

 $Q =$ 

$P^Q =$