Lecture XXXII: Duality, antichains, order ideals & graded posets

Last Time: · defined posets, weak & induced subposets, intervals, order preserving maps. (→ determine a topology on P)
· Drew finite posets via Hasse diagrams (via covering relations)
· Build new posets from old, $+ = \sqcup$, $\times$, $\oplus$, $\otimes$, power sets → Hasse diagram rules

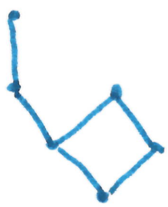
§1 More new posets from old:

① Def: A poset P has a $\hat{0}$ if there exists an element $\hat{0}$ of P s.t. $\hat{0} \leq t$ $\forall t \in P$

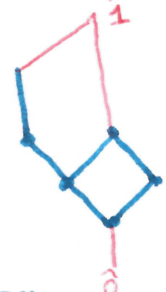
. ——— $\hat{1}$ ——— $\hat{1}$ ——— $t \leq \hat{1}$ ———.

Write $\hat{P}$ = poset obtained from P by artificially adding a $\hat{0}$ & a $\hat{1}$ to P (independently of P having already a $\hat{0}$ and/or a $\hat{1}$)

Example:

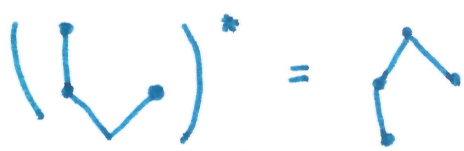
P = 

has $\hat{0}$, no $\hat{1}$

⇝ $\hat{P}$ = 

(now $\hat{0} < \hat{1}$)

② Def: The dual of P is the poset P^* where $\begin{cases} \cdot\ |P^*| = |P| \\ \cdot\ x \leq_{P^*} y \iff y \leq_P x \end{cases}$

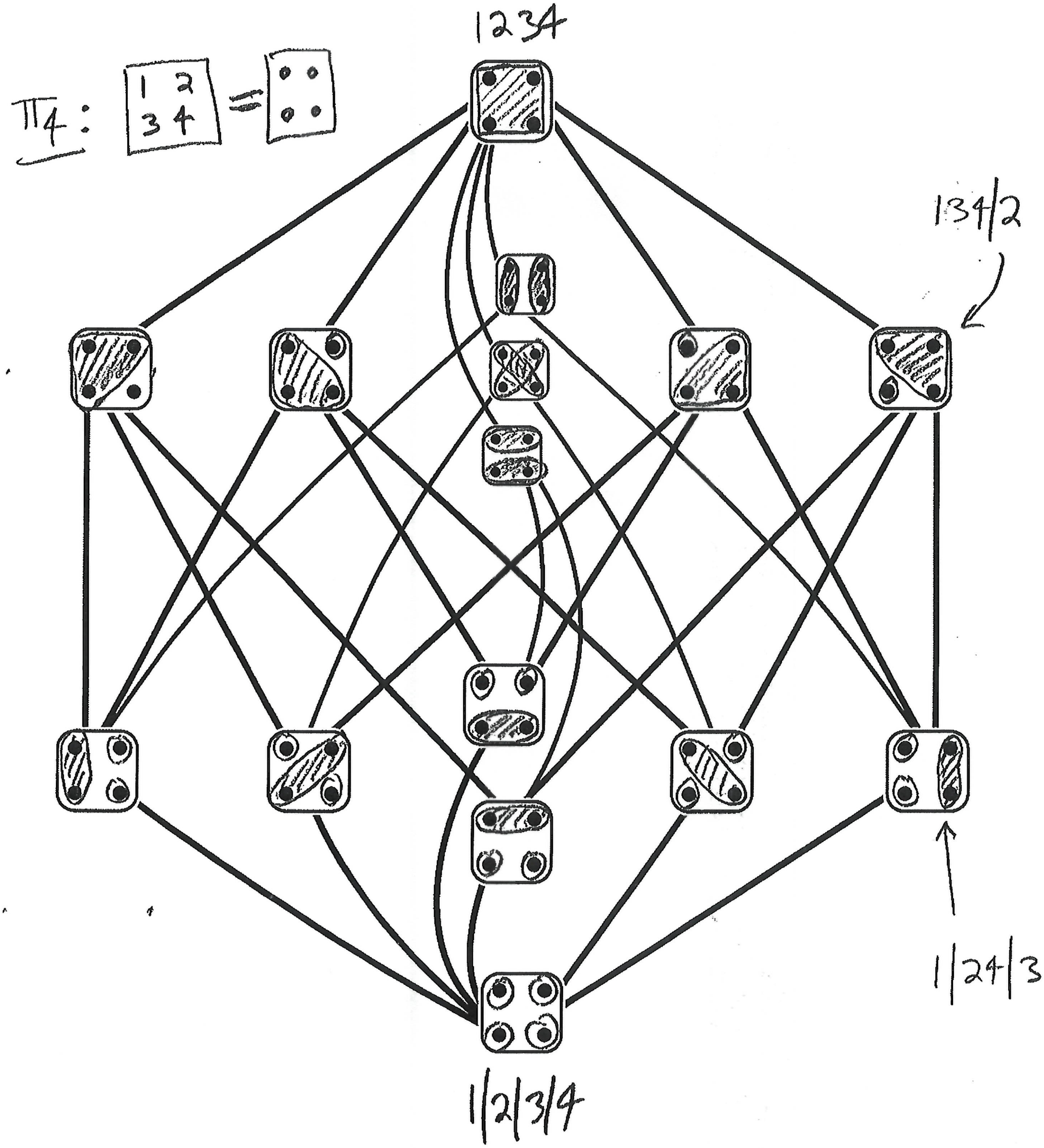
Example:  (flip Hasse diagram)

· A poset is self-dual if $P^* \simeq P$

Posets	n	B_n	Π_n	$B_n(q)$	D_n
$\hat{0}$	1	$\emptyset$	$1\|\cdots\|n$	$<0>$	1
$\hat{1}$	n	$[n]$	$1\cdots n$	$\mathbb{F}_q^n$	n
Self-dual	YES	YES	NO ($n=4$) (*)	YES	YES
Isomorphism $P \simeq P^*$	$i \mapsto n-i+1$	$S \mapsto [n] \setminus S$	—	$w \mapsto w^\perp$ for $<,>$ standard bilin. form $<e_i, e_j> = \delta_{ij}$	$d \mapsto \frac{n}{d}$

(*) elements covering $\hat{0}$ are covered by 3 elements each, but elements covered by $\hat{1}$ don't cover 3 elements each (some cover 2!)

ALSO: 6 elements cover $\hat{0}$ but $\hat{1}$ covers 7 elements in Π_4


π4 :
1 2
3 4
=
1234
134|2
1|24|3
1|2|3|4

③. Given 2 posets P, Q we say that P <u>refines</u> Q if $|P| = |Q|$ & Q is a weak subposet of P. Equivalently $id: Q \longrightarrow P$ is order preserving.

<u>Obs</u>: Every poset can be refined to a <u>total order</u> ("Order Extension Principle") (Proof via Axiom of Choice (Marczewski)) $\exists f: P \to$ TOTAL order on P = <u>chain</u>

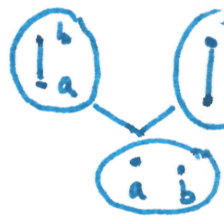
In particular, if $|P| = n$, then $\exists f: P \xrightarrow{bij} n$ = n-chain order-preserving

<u>Name</u>: Linear extension or Topological sorting of P.

Number of linear extensions of $P =: e(P)$ measures "complexity" of P.

. Given S: all poset structures on S is a poset under refinement.

<u>Ex</u>: $S = \{a, b\}$



§ 2 Chains, multichains & antichains:

<u>GOAL</u>: Study <u>internal</u> properties of P by studying subposets of P isomorphic to n-chains

<u>Def</u>: . Given $x, y \in P$, a <u>chain</u> from x to y of <u>length l</u> is a collection

$$C := \quad x =: x_0 < x_1 < x_2 < \dots < x_l := y$$

[$(l+1)$ elements for a chain of length l]

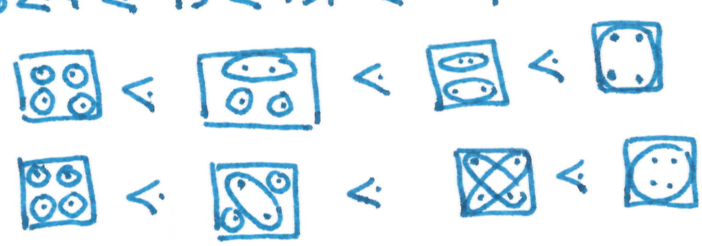
. A chain of length l is <u>saturated</u> (or unrefinable) if $x_i \lessdot x_{i+1}$ (covers) $\forall i = 0, \dots, l-1$

($\nexists u \in [x,y] \setminus C$ st. $C \cup \{u\}$ is a chain)

. C is <u>maximal</u> if it is saturated and runs from a minimal element of P ($s \in P$: $\nexists t \in P$ with $t < s$) to a maximal one ($s \in P$ | $\nexists t \in P$ with $s < t$)

<u>Example</u>: In B_4: $\emptyset \lessdot 1 \lessdot 13 \lessdot 134 \lessdot 1234$

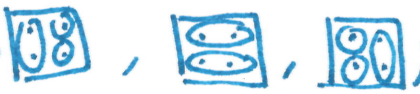
In Π_4:



<u>Def</u>: A <u>multichain</u> is a chain with repeated elements (multiset whose underlying set is a chain of P). length $:= \# C - 1$

<u>Ex</u>: $\emptyset \leq \emptyset \leq 12 \leq 124 \leq 124$ is a length 4 multichain in B_4

Def. An <u>antichain</u> (or Sperner family) is a subset A of a poset P where any 2 elements in A are incomparable.

<u>Ex</u>: {  } in Π_4

Def. An <u>order ideal</u> (or <u>semi-ideal</u>) of P is a subset I of P s.t.:

$t \in I$ & $s \leq t \Longrightarrow s \in I$ (look below t in Hasse diagram to get $\Lambda_t = \langle t \rangle$ = ideal gen by t = ~~principal~~ principal order ideal gen by t)

- A <u>dual order ideal</u> (or <u>filter</u>) of P is an order ideal of P^* ($t \in I$ & $s \geq t \Rightarrow s \in I$) ($V_t = \{s \in P \mid s \geq t\}$ ppal dual order ideal)

<u>Prop</u>: Assume P is finite. Then

$$\{\text{antichains of } P\} \xleftrightarrow{1:1 \text{ to } 1} \{\text{order ideals of } P\}$$
$$A = \{\text{max'l elements of } I\} \longleftarrow I$$
$$A \longmapsto I := \{s \in P : s \leq t \text{ for some } t \in A\}$$
(A generates I)

<u>Remark</u>: $J(P)$:= set of all order ideals of P. Then, $J(P)$ is a poset under $\subseteq$.

<u>§3 Graded posets</u>:

<u>Def</u>: The <u>length</u> or <u>rank</u> of a finite poset P is:

$$\ell(P) = \max\{\ell(C) : C \text{ chain of } P\}$$

If $P = [s,t]$, write $\ell(s,t) := \ell([s,t])$

<u>Def</u>: If every max'l chain of P has the same length ($=n$), we say that P is <u>graded</u> of <u>rank n</u>

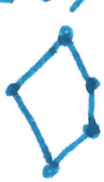
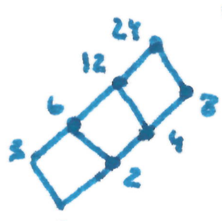
If so, we can define a <u>rank</u> function $rk = \rho : P \to \{0, \dots, n\}$ by

$$\left.\begin{array}{ll} \rho(s) = 0 & \text{if } s \text{ minimal} \\ \rho(t) = \rho(s) + 1 & \text{if } t \gtrdot s. \end{array}\right\} \quad \rho(t) = \ell(C) \quad C \text{ saturated chain from a minimal element of } P \text{ to } t$$

(well-def because P is graded)

<u>Obs</u>: If $s \leq t$, write $\rho(s,t) = \rho(t) - \rho(s) = \ell(s,t)$

- If $\rho(s) = i$, we say s has <u>rank i</u> ⇝ Draw Hasse diagram with "levels" by rank

<u>Example</u>:  not graded ; D_{24}:  is graded $\rho(D_{24}) = 4$

$\rho(4) = 2$

Def If P is a graded poset of rank n its <u>rank generating function</u> is

$$F(P,x) = \sum_{p\in P} x^{\mathrm{rank}(p)} = \sum_{i=0}^{n} p_i(P)\, x^i$$

$p_i(P) = \#\{s \in P \mid rk(s) = i\}$ i^{th} Whitney # of the 2^{nd} kind.

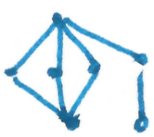
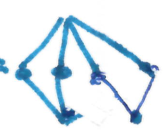
Ex: $F(D_{24}, x) = 1 + 2x + 2x^2 + 2x^3 + x^4$

$= (1+x+x^2+x^3)(1+x)$

$= F(4 =$  $, x)\ F(2 =$  $, x)$

Consistent with $D_{24} \simeq 4 \times 2$. (next time)

Obs: $F(P,x)$ doesn't determine P

$P =$  & $Q =$  are graded posets of rank 2, <u>non</u>-isomorphic

but $F(P,x) = F(Q,x)$.