Lecture XXXIII: More on graded posets; lattices

§1 More on graded posets

Recall: For P finite poset, we say P is graded if every maximal chain has the same length $(= \text{rank}(P) = n)$

We define a rank function on P: $\rho = \text{rk} : P \longrightarrow \{0, \ldots, n\}$
$t \longmapsto \ell(C)$ for C max. chain from ~~a~~ min'l in P to t.

$\rightsquigarrow$ Generating function $F(P,x) = \sum_{t \in P} x^{\text{rk}(t)} = \sum_{i=0}^{n} \boxed{P_i(P)}\, x^i$

ith Whitney # of the 2nd kind

Note: $P \simeq Q$ graded (finite) posets $\Rightarrow F(P,x) = F(Q,x)$

But $\not\Leftarrow$: $P =$  $\not\simeq$  $= Q$ $\quad F(P,x) = F(Q,x) = 2 + 3x^2 + x^3$

Obs: Can extend definitions to certain infinite posets

$P = \bigsqcup_{i \in I} P_i$ graded if every max'l chain in P has the form:

$t_0 < t_1 < t_2 < \cdots$ where $t_i \in P_i$

$\rightsquigarrow \exists\ \rho : \text{rk} : P \longrightarrow \mathbb{N}_0$ & $F(P,x) \in \mathbb{C}[[x]]$

Prop: P, Q graded posets of rank n & m. Then:

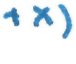
(1) $P \times Q$ is graded of rank $m+m$ & $F(P \times Q, x) = F(P,x)\,F(Q,x)$
↳ ($\leq$ comp by comp) (max'l chain = prod of max'l chains)

(2) $P + Q = P \sqcup Q$ —— iff $n = m$. If so $\text{rk} = n$ & $F(P \sqcup Q, x) = F(P,x) + F(Q,x)$
↳ (no comparison b/t P & Q)

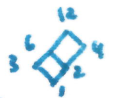
(3) $P \otimes Q$ is graded of rank $m + n(m+1)$ & $F(P \otimes Q, x) = F(P, x^{m+1})\, F(Q,x)$
↳ $(p,q) \leq (p',q') \Leftrightarrow (p < p')$ or $(p = p'$ & $q \leq q')$

Q: Is P^Q also graded? ($F(P^Q, x)$? Is it hard to compute?)

(4) P^* is graded of rank n & $F(P^*, x) = x^n\, F(P, \frac{1}{x})$

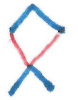
Examples: ① $F($  $, x) = F($  $, x) + F($  $, x)$

$3 + 2x = (2+x) + (1+x)$



② $F(D_{12}, x) = F($  $, x)\; F($  $, x)$

$1 + 2x + 2x^2 + x^3 = (1 + x + x^2)(1 + x)$

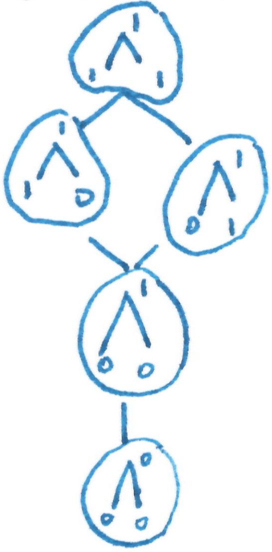
③ $F(1 \otimes \Lambda; x) = F($  $, x) = 2+x+2x^2+x^3$

$F(1, x^2)\ F(\Lambda, x) = (1+x^2)(2+x)$ $\overset{\checkmark}{=}$

④ **Ex:** (YES) $P = |$, $Q = \Lambda$ $\Rightarrow$ $P^Q =$

rk 1 rk 1

rk 3 graded



$F(P^Q, x) = 1+x+2x^2+x^3$

Counter-examples? Proof?

④ $P = \Lambda$ $\quad P^* = Y$

$F(P,x) = 2+x+x^2$ $\quad F(P^*, x) = 1+x+2x^2 = x^2\left(2+\frac{1}{x}+\frac{1}{x^2}\right)$ ✓

§2 Examples

Poset	n	B_n	Π_n	$B_n(q)$	D_n $n = p_1^{a_1}\cdots p_k^{a_k}$
$rk(t)$ $t \in P$	$t-1$	$\lvert t \rvert$	$n-k$ if $t = B_1\lvert\cdots\rvert B_k$	$\dim(t)$	$rk(p_1^{l_1}\cdots p_k^{l_k}) = l_1+\cdots+l_k$
rank(P)	$n-1$	n	$n-1$	n	$\sum_{1\le i\le k} a_i$
$p_i(P)$	1	$\binom{n}{i}$	$\boxed{S(n,n-i)}$	$\left[{n \atop i}\right]_q$	messy
$F(P,y)$	$\sum_{i=0}^{n-1} y^i = [n]_y$	$\sum_{i=0}^{n} \binom{n}{i} y^i = (1+y)^n = [2]_y^n$	$\sum_{i=0}^{n} S(n,i)\, y^i$	$\sum_{i=0}^{n} \left[{n \atop i}\right]_q y^i$	$\prod_{i=1}^{k} [a_i+1]_y$

$D_n \cong \overset{\text{chain}}{(a_1+1)} \times \cdots \times \overset{\text{chain}}{(a_k+1)}$ (primes are independent!)

Def $= S(n,k) = \left\{ {n \atop k} \right\}$ = Stirling number of the 2nd Kind ($\#\{$part of $[n]$ into k blocks$\}$)

Ex: $S(4,1) = S(4,4) = 1$, $S(4,2) = 7$, $S(4,3) = 6$

Note. $[n] = B_1 \sqcup \cdots \sqcup B_k \Rightarrow \lvert B_1 \rvert, \ldots \lvert B_k \rvert$ gives (after reordering) a partition of n of length k.

Recurrence Relations:

- $S(n+1,k) = k\,S(n,k) + S(n,k-1)$ for $0<k<n$
  - ↳ place $(n+1)$ in one of the k blocks
  - ⟶ $(n+1)$ its a single block (ISOLATED)
- $S(0,0) = 1$, $S(n,0) = S(0,n) = 0$ for $n>0$

Triangle

$n \backslash k$	0	1	…
0	1		0
1	0	1	
2	0	1	
3	⋮	⋮	
4	⋮	⋮	

§3. Lattices:

Motivation: In many posets, we can take "intersections" & "sums/unions" of pairs of elements.

Ex ① B_n : $T, S \subseteq [n] \rightsquigarrow T \cap S$ & $T \cup S$

② $B_n(q)$: $V, W \subseteq \mathbb{F}_q^n \rightsquigarrow V \cap W$ & $V + W$

③ D_n : $a, b \mid n \rightsquigarrow \gcd(a,b)$ & $\operatorname{lcm}(a,b)$ both divide n.

In posets, we view these operations as greatest lower bound & least upper bound:

Def: A poset L is a lattice if the following two conditions hold:

(i) For any $x, y \in L$, x & y have a greatest lower bound or meet $= x \wedge y$

(ii) ———————————— least upper bound or join $= x \vee y$

Def: • Least upper bound of $x, y \in P$ = $u \in P$ st • $u \geq x$, $u \geq y$
• $\forall v \in P$ with $v \geq x$ & $v \geq y \Rightarrow v \geq u$

• greatest lower bound of $x\, y \in P$ = $u \in P$ st • $u \leq x$, $u \leq y$
• $\forall v \in P$ with $v \leq x$ & $v \leq y$ $\Rightarrow v \leq u$.

In particular: (1) $x \wedge y \leq x$, $x \wedge y \leq y$
$\forall v : v \leq x, y \Rightarrow v \leq x \wedge y$ } ($x \wedge y$ is unique!)

(2) $x \vee y \geq x$, $x \vee y \geq y$
$\forall v$: $x \leq v$, $y \leq v \Rightarrow x \vee y \leq v$ } ($x \vee y$ is unique!)

Def: A sublattice is a subposet of L closed under $\vee$ & $\wedge$.

Ex ④ Π_n : meets = "common refinements"
joins = set partitions with "least required common mergings"

$\pi = 125 \mid 38 \mid 4 \mid 67 \mid 9$ $\rightsquigarrow$ $\pi \wedge \mu = 12 \mid 3 \mid 4 \mid 5 \mid 67 \mid 8 \mid 9$

$\sigma = 124 \mid 3 \mid 679 \mid 5 \mid 8$ $\quad$ $\pi \vee \mu = 1245 \mid 38 \mid 679$