Lecture 34: More on lattices

§1 Basic properties of lattices

Recall: A **lattice** L is a poset with 2 operations.

$$\wedge = \text{meet} : L \times L \longrightarrow L \quad \& \quad \vee = \text{join} : L \times L \longrightarrow L$$

satisfying:
- $x \wedge y \leq x, y$
- $\forall v$ with $v \leq x, y \Rightarrow v \leq x \wedge y$ (greatest lower bound)

- $x \vee y \geq x, y$
- $\forall v$ with $v \geq x, y \Rightarrow v \geq x \vee y$ (least upper bound)

- **Sublattice**: subposet of L closed under $\wedge$ & $\vee$.

Examples: (1) B_n : $\wedge = \cap$ & $\vee = \cup$

(2) $B_n(q)$ = $\wedge = \cap$ & $\vee = +$

(3) D_n = $\wedge = \gcd$ & $\vee = \operatorname{lcm}$

(4) Π_n : $\wedge$ = common refinement, $\vee$ = "least required common merging".

Prop 1: $\vee, \wedge : L \times L \longrightarrow L$ for L lattice satisfy:

(1) Associative, commutative

(2) Idempotent : $t \wedge t = t \vee t = t \quad \forall t \in L$

(3) Absorption Laws: $s \wedge (s \vee t) = s = s \vee (s \wedge t)$

(4) $s \wedge t = s \iff s \vee t = t \iff s \leq t$

Prop 2: (1) All finite lattices have $\hat{0}$ & $\hat{1}$

(2) L, M lattices $\Rightarrow L^*$, $L \times M$, $\widehat{L+M}$ are lattices

Proof: (1). If x, y are both maximal & $x \neq y \underset{\text{Prop 1}}{\Longrightarrow} x \vee y \gneq x, y$, so x & y are not maximal. Contr!

(maxl element exists for finite posets
" infinite " by Zorn's Lemma)

- $\exists$ maxl element $= \bigvee_{x \in L} x \geq y \quad \forall y \in L \Rightarrow \hat{1} = \bigvee_{x \in L} x$.

(induct on $|L|$ + use associativity)

Analogous statement for $\hat{0}$ via meet.

(2). $\wedge_{L^*} = \vee_L$ & $\vee_{L^*} = \wedge_L$

- $\wedge_{L \times M} = \wedge_L \times \wedge_M$; $\vee_{L \times M} = \vee_L \times \vee_M$
- For $L+M$ to be a lattice, we must add $\hat{1}$ & $\hat{0}$ since
  - $l \vee m = \hat{1}$,
  - $l \wedge m = \hat{0}$.

Q: How to check a poset is a lattice? Often meets are easier than joins.

Def: A poset P is a <u>meet-semilattice</u> if $\forall x, y \in P$: $\exists\, x \wedge y$ (meets exist)

——— join-semilattice ——— $\exists\, x \vee y$ (joins ——)

Prop 3: Assume P is a finite meet-semilattice. Then:

P has a $\hat{1}$ $\iff$ P is a lattice.

Duality gives the statement for join-semilattices P: P has a $\hat{0}$ $\iff$ P is a lattice.

Proof $\Rightarrow$) Pick $s, t \in P$ & define $S = \{u \in P \mid u \geq s,\ u \geq t\}$

- $S \neq \emptyset$ because $\hat{1} \in P$
- S is finite

Claim: $s \vee t = \bigwedge_{x \in S} x$. (induction on $|S|$ + associativity says this meet exists).

($\Leftarrow$) done before. ∎

⚠ Statement fails for infinite posets since for $|S| = \infty$ we don't know that $\bigwedge_{x \in S}$ exists! The next definition asses when it holds:

Def. If every subset of L has a meet & join, then we say L is a <u>complete lattice</u>.

In particular, complete lattices always have $\hat{0}$ & $\hat{1}$.

§2 Graded Lattices

Obs: Most interesting lattices for enumerative combinatorics are <u>graded</u>.

Q: How do $\wedge$, $\vee$ & rank functions $\rho: L \to \mathbb{Z}_{\geq 0}$ interact?

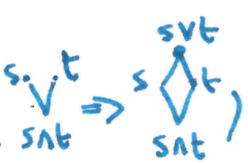
Example: (1) B_n: $\rho(T) = |T|$ & $\rho(T \cap S) + \rho(T \cup S) = \rho(T) + \rho(S)$

(2) $B_n(q)$: $\rho(V) = \dim V$ & $\rho(V \wedge W) + \rho(V + W) = \rho(V) + \rho(W)$

In general, we expect inequalities, rather than $=$.

Proposition 4: Assume L is a finite lattice. TFAE.

(1) L is graded & the rank function ρ satisfies:
$$\rho(s) + \rho(t) \geq \rho(s \vee t) + \rho(s \wedge t) \quad \forall s, t \in L$$

(2) If s & t both <u>cover</u> $s \wedge t$, then $s \vee t$ covers both s & t. 

Def: A finite lattice satisfying (1) or (2) is called an <u>upper semimodular</u> lattice.

Dual Prop 4: Assume L is a finite lattice. TFAE.

(1') L is graded & ρ satisfies:

$$\rho(s) + \rho(t) \leq \rho(s \vee t) + \rho(s \wedge t) \quad \forall s,t \in L$$

(2') If $s \vee t$ covers both s & t, then s & t both cover $s \wedge t$.

Def: A finite lattice satisfying (1') or (2') is called <u>lower-semimodular</u>.

Def: A <u>modular lattice</u> is a lattice that is both upper- & lower-semimodular.

(Ex: B_n, $B_n(q)$)

Prop 5: A lattice is modular if and only if it does not contain a sublattice isomorphic to 

Proof of Prop 4:

<u>(1) ⟹ (2)</u> Assume $s \wedge t \lessdot s, t$. Then $\rho(s) = \rho(s \wedge t) + 1 = \rho(t)$

Moreover $s \vee t \gneq s, t$ by Prop 1(4). In particular $\rho(s \vee t) > \rho(s) = \rho(t)$

By (1): $\rho(s \vee t) + \rho(s \wedge t) = \rho(s \vee t) + \rho(s) - 1 \leq 2\rho(s)$

gives $\rho(s \vee t) \leq \rho(s) + 1$

So $\rho(s) < \rho(s \vee t) \leq \rho(s) + 1$ all in $\mathbb{Z}$ $\Rightarrow \rho(s \vee t) = \rho(s) + 1 = \rho(t) + 1$

So $s, t \lessdot s \vee t$.

<u>(2) ⟹ (1)</u> To show Ⓐ L is graded

Ⓑ inequality on ranks.

Ⓐ Argue by contradiction. Assume L is not graded, and let $[u,v]$ be an interval in L of minimal length that is not graded.

Then: we have $s_1, s_2 \in [u,v]$ covering u with all mxl chains in $[s_i, v]$ ↳(mxl length of chain in the set) have length l_i for $i=2$ (they are graded!) <u>but</u> $l_1 \neq l_2$.

But, $u = s_1 \wedge s_2$ since $u \lessdot s_1, s_2$. So, by (2) $s_1 \vee s_2$ covers both s_1 & s_2.

In particular, we can build saturated chains in $[s_i, V]$ of the form:

$$s_i \lessdot s_1 \vee s_2 \lessdot t_1 \lessdot \cdots \lessdot t_k = V. \quad \text{where } k = rk([s_1 \vee s_2, V])$$

So $\ell_1 = k+1 = \ell_2$ contradiction!

$[s_1 \vee s_2, V]$ is graded because $\ell([u,v]) > \ell([s_1 \vee s_2, V])$

(B) Again, we argue by contradiction, i.e. there are $s, t \in L$ satisfying

$$(*) \quad \rho(s) + \rho(t) < \rho(s \vee t) + \rho(s \wedge t)$$

Pick a pair (s,t) satisfying $(*)$ and:

(i) $\ell([s \wedge t, s \vee t])$ minimal

(ii) $\rho(s) + \rho(t)$ minimal among those satisfying (i) & $(*)$.

By (2) we know that $s, t \not\gtrdot s \wedge t$, otherwise $s, t \lessdot s \vee t$

& $\rho(s) + \rho(t) = (\rho(s \wedge t) + 1) + (\rho(s \vee t) - 1) \Rightarrow (*)$ fails.

WLOG, assume $s \wedge t \not\lessdot s$, so $\exists\, s' \in L$ with $s \wedge t < s' < s$

By the minimality conditions, (s', t) doesn't satisfy $(*)$

- $\ell([s \wedge t, s \vee t]) \geq \ell([s' \wedge t, s' \vee t])$ with $s' \wedge t = s \wedge t$, $s' \vee t \leq s \vee t$
- $\rho(s') < \rho(s)$ so $\rho(s') + \rho(t) < \rho(s) + \rho(t)$

$$(**) \quad \rho(s') + \rho(t) \geq \underbrace{\rho(s' \vee t)}_{\leq\, \rho(s \vee t)} + \underbrace{\rho(s' \wedge t)}_{=\, \rho(s \wedge t)}$$

Now: $\rho(s') < \rho(s)$ & $\rho(s \wedge t) = \rho(s' \wedge t)$

$$(*) \Rightarrow \rho(s) < \rho(s \vee t) + \rho(s \wedge t) - \rho(t)$$

$$+ \quad (**) \Rightarrow \rho(s' \vee t) \leq \rho(s') - \rho(s' \wedge t) + \rho(t)$$

$$\boxed{\rho(s) + \rho(s' \vee t) < \rho(s \vee t) + \rho(s')}$$

Notice $s \wedge (s' \vee t) \geq s'$ (Since $s' \leq s' \vee t$ & $s' \leq s$)

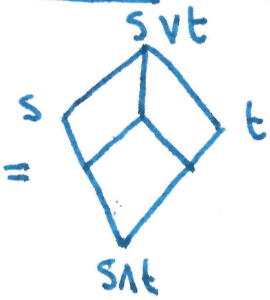
$s \vee (s' \vee t) = s \vee t$ by associativity.

Set $U = s$, $T = s' \vee t$. Then, U & T satisfy $(*)$ since:

$$\rho(U) + \rho(T) < \rho(U \vee T) + \rho(s') \leq \rho(U \vee T) + \rho(U \wedge T)$$

but $\ell([\underbrace{U \wedge T}_{\geq\, s' \,>\, s \wedge t}, \underbrace{U \vee T}_{=\, s \vee t}]) < \ell([s \wedge t, s \vee t])$, contradicting the minimality of s & t.

Example: Upper- but **NOT** lower-semimodular lattice. In particular, not modular

$L =$ 

$$\left.\begin{array}{l} \bullet \text{ We have } s\vee t \gtrdot s \ \& \ s \vee t \gtrdot t, \\ \bullet \ s \not\gtrdot s \wedge t \end{array}\right\} \Rightarrow L \text{ \underline{not} lower semimodular}$$

- Upper-semimodular lattice by condition (2).