Lecture XXXV Distributive Lattices

Recall: A finite lattice L is upper-semimodular if it satisfies the following equivalent conditions

(1) L is graded & the rank function $\rho: L \to \mathbb{Z}_{\geq 0}$ satisfies:

$$\rho(s) + \rho(t) \geq \rho(s \wedge t) + \rho(s \vee t) \quad \forall s,t \in L \qquad (*)$$

(2) If $s \wedge t \lessdot s,t$ then $s,t \lessdot s \vee t$

Proof $2 \Rightarrow 1$ from Lecture XXXIV

Obs: L is lower-semimodular if L^* is upper-semimodular.

· L is modular if it is both lower & upper-semimodular. (~~se~~ $=$ holds in $(*)$)

[HW7] A finite lattice L is modular if and only if it does not contain  as a sublattice

Prop 1: A finite lattice L is modular if and only if $\forall s,t,u \in L$ with $s \leq u$ we have $\boxed{s \vee (t \wedge u) = (s \vee t) \wedge u}$

Consequences: · A sublattice of a modular finite lattice is also modular.

· Can use Prop to define infinite modular lattices.

§1. Distributive Lattices:

Def: For a lattice L, the distributive laws are:

(1) $s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u)$,

(2) $s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$.

Remark: (1) holds $\iff$ (2) holds.

Example: ① B_n: $W, T, S \subseteq [n]$ $\quad S \cap (T \cup W) = (S \cap T) \cup (S \cap W)$

② n-chain: s, t, u Easy check; eg if $s \leq t \leq u$: $s \vee (\underbrace{t \wedge u}_{=t}) = t = (\underbrace{s \vee t}_{=t}) \wedge (\underbrace{s \vee u}_{=u})$

(5 more relative orderings are analogous)

Non-example: $B_n(q)$: $n = 2$ & $q = 2$: $\mathbb{F}_2^2 = \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

$W = \langle \begin{bmatrix}1\\0\end{bmatrix} \rangle$, $V = \langle \begin{bmatrix}0\\1\end{bmatrix} \rangle$, $Y = \langle \begin{bmatrix}1\\1\end{bmatrix} \rangle$. Then: $W \cap (V + Y) = W \cap \mathbb{F}_2^2 = W$

$W \cap V + (W \cap Y) = \{\begin{bmatrix}0\\0\end{bmatrix}\} + \{\begin{bmatrix}0\\0\end{bmatrix}\} = \{\begin{bmatrix}0\\0\end{bmatrix}\}$

Obs: Distributive Laws distinguish B_n from $B_n(q)$.

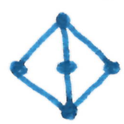
Def: A lattice L is distributive if it satisfies the distributive laws.

Prop 2: L distributive $\Rightarrow$ L modular (In particular, it is graded!)

Proof: Use (1) & assume $s \leq u$. Then, $s \vee u = u$. In particular,

$$s \vee (t \wedge u) = (s \vee t) \wedge \underbrace{(s \vee u)}_{= u} \qquad \text{for all } s,t,u \in L \text{ with } s \leq u$$

Thus, Prop 1 gives: L is modular. ∎

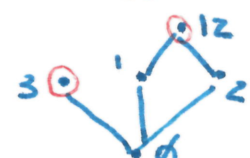
Prop 3: L is distributive if and only if L is modular and has no sublattice isomorphic to 

§3. Fundamental Thm of finite Distributive Lattices (FTFDL):

Recall Order ideal I of a poset P is a subset satisfying:

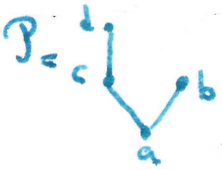
if $t \in I$ & $s \leq t$ $\Rightarrow$ $s \in I$

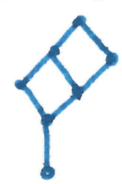
Ex: B_3: $I = \langle 3, 12 \rangle$ =

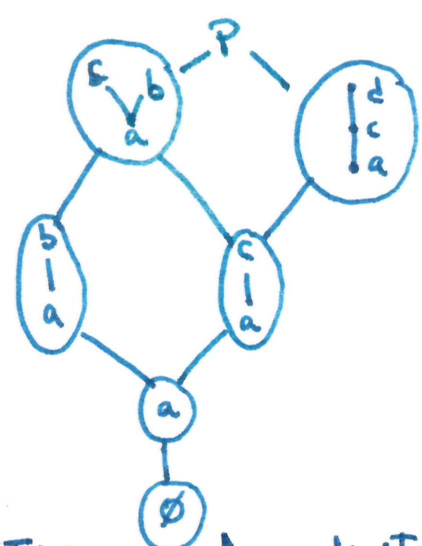


. Finite P: $\{\text{order ideals of } I\} \xleftrightarrow[\text{1-to-1}]{} \{\text{antichains of } P\}$

Def: $J(P)$ = poset of order ideals of any poset P, ordered by inclusion.

Ex: $P =$ 

$\rightsquigarrow$ $J(P) \simeq$  where



. graded poset with rank function $\rho(I) = |I|$

. subposet of 2^P

. **Notice:** distributive b/c it avoids both  (as sublattices)

FTFDL: Any finite distributive lattice is isomorphic to $J(P)$ for a unique (up to iso) poset P (i.e. $J(P) \simeq J(Q) \Rightarrow P \simeq Q$)