Lecture XXXVI: Finite distributive lattices & geometric lattices

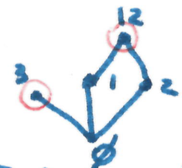
§1 Classification of finite distributive lattices:

Recall: $I \subset P$ (poset) is an order ideal if:

(1) I is a sub poset of P

(2) $t \in I$ & $s \leq t \Rightarrow s \in I$.

Eg: $B_3 \supseteq I = \langle 3, 12 \rangle \rightsquigarrow I$:



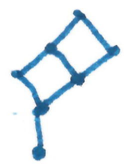
Prop 1: $J_{(P)} := \{$ order ideals of $P\} \xleftrightarrow{1\text{-to-}1} \{$ antichains of $P\}$ if P is finite.

$\rightsquigarrow J(P)$ is a subposet of $\{0,1\}^P$ (ordered by inclusion)

(*) (also of $\underline{2}^P$ ($f_I : P \to \{0,1\}^{<}$) $f_I(x) = 0 \Leftrightarrow x \in I$ ord-ideal)
$I \subseteq I' \Leftrightarrow f_I \geq f_{I'}$

Ex: $P =$

$\rightsquigarrow$



$= J(P) \subseteq 2^3$ graded with $\rho(I) = |I|$)

Prop 2: $J(P)$ is a lattice with $\wedge =$ intersection & $\vee =$ union. So $J(P)$ is a sublattice of $\{0,1\}^P$, in particular, distributive (so graded with $\rho(I) = |I|$)

Proof: • Easy check (by definition!): I, J order ideals of $P \Rightarrow I \cap J$, $I \cup J$ are order-ideals of P.

• $\{0,1\}^P$ is a distributive lattice & $J(P)$ is a sublattice $\Rightarrow J(P)$ is distrib

$\{0,1\}^P$ is graded with $\rho(I) = |I|$. $J(P)$ has the same rank function as $\{0,1\}^P$ because $\mathrm{rk}(J(P)) = |P|$ (Take $\tilde{P}$ linear extn of P & $s_1 < s_2 < \ldots < s_k$ the maxl chain in $\tilde{P} = (P, \leq)$ total, $\boxed{k = |P|}$
$\Rightarrow \cancel{\emptyset} < \langle s_1 \rangle < \langle s_2 \rangle < \langle s_3 \rangle < \cdots < \langle s_k \rangle = P$ is maxl chain in $J(P)$)

Cor: $J(P)^* \simeq 2^P \simeq J(P^*)$ (*)

FTFDL: Assume L is a finite distributive lattice. Then, there is a unique (up to iso) poset P such that $L \simeq J(P)$

Proof strategy: Build candidate P, show $L \simeq J(P)$. Then, prove uniqueness.

Def: Fix L a distributive lattice. A join-irreducible element $s \in L$ is one satisfying (1) $s \neq \hat{0}$

(2) $s \neq t \vee u$ for any $t, u \lneq s$

$\mathcal{J}(L) = \{ s \in L \mid s$ is join-irreducible$\}$

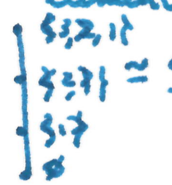
(Duality, define meet-irreducibles: (1) $s \neq \hat{1}$ (2) $s \neq t \wedge u$ for any $t, u \gneq s$.)

Lemma: If L is finite, then $s \in L$ is join-irred $\Leftrightarrow$ s covers exactly one element.

Ex: ① $L = \mathbf{4}$

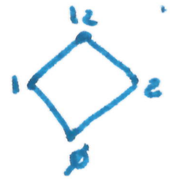
$\mathcal{J}(\mathbf{4}) = \{1,2,3\}$
$\subseteq L$ subposet

$J(\mathcal{J}(\mathbf{4})) =$



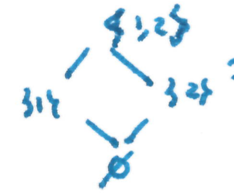
$\simeq \mathbf{4}$.

② $L = B_2$



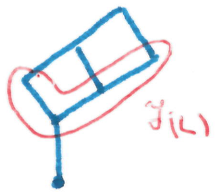
$\mathcal{J}(B_2) = \{1,2\}$

$J(\mathcal{J}(B_2)) =$



$\simeq B_2$

③ $L =$



$\mathcal{J}(L) =$

& $J(\mathcal{J}(L)) \simeq L$ (page 1)

Def: An order ideal I of a finite poset P is join-irreducible in $J(P)$ (as an element of $J(P)$)
$\Leftrightarrow$ $I = \langle a \rangle$ for some $a \in P$

Thus: $\mathcal{J}(J(P)) = \{$join-irred ideals of $J(P)\} \xleftrightarrow{1-1} P$. (order-preserving?)

Lemma: If P is finite, then $\mathcal{J}(J(P)) \simeq P$.

Proof: $\langle s \rangle \subsetneq \langle t \rangle \Leftrightarrow s \lneq t$. (order-preserving & inverse also order preserving!)

Corollary: $J(P) \simeq J(Q)$ for P, Q finite posets $\Leftrightarrow P \simeq Q$.

Proof of FTFDL: Set $P = \mathcal{J}(L)$. T.S. $L \simeq J(P)$ (uniqueness follows by Corollary)

Define $\phi : L \longrightarrow J(P)$
$t \longmapsto I_t := \{s \in P \mid s \leq t\}$ (well-def & order-preserving)

- ϕ injective: L lattice & $t = \bigvee_{x \in I_t} x$.
- ϕ surjective: Pick $I \in J(P)$ & set $t := \bigvee_{x \in I} x$.
  - $I_t \supseteq I$: by def.
  - $I_t \subseteq I$: Pick $u \in I_t$ & show $u \in I$.

Note: $t = \bigvee_{s \in I} s = \bigvee_{s \in I_t} s \underset{\substack{- \wedge u \\ + \text{distrib laws}}}{\Longrightarrow} \bigvee_{s \in I} (s \wedge u) = \bigvee_{s \in I_t} (s \wedge u) = u$ because $u \in I_t$ & $s \wedge u \leq u$ $\forall s \in I_t$

Key fact: $u \in P \Rightarrow u$ is join-irreducible. So in (LHS), some $s \in I$ satisfies $s \wedge u = u$. In particular $u \leq s$ for $s \in I \underset{I \text{ order-ideal}}{\Longrightarrow} u \in I$.

Conclude: L finite & ϕ: order-preserving bijection $\underset{[HW7]}{\Longrightarrow}$ ϕ is isomorphism of posets. ☐

- Classification extends to finitary distrib. lattices $\equiv$ locally finite, distrib. lattices with $\hat{0}$

  ⇝ L has a unique rank function $\rho: L \to \mathbb{N}_{\geq 0}$
  $t \mapsto \ell(C)$ C = any sat. chain from $\hat{0}$ to t.

Prop: Fix P poset where every principal order ideal ($\langle a \rangle$) is finite. Then, $J_f(P) = \{$finite order-ideals of $P\}$ with $\leq = \subseteq$ is a finitary distrib. lattice. Conversely, if L is a finitary distr. lattice & $P = \mathcal{J}(L)$ (subposet of join-irred), then every ppal order ideal of P is finite & $L \simeq J_f(P)$.

Ex: $P = \mathbb{N}_0 \times \mathbb{N}_0$ ⇝ $J_f(P)$ = Young Lattice $\simeq \{\lambda \vdash n, n \in \mathbb{N}\}$ ordered by containment of Young diagrams.

§2 Geometric Lattices:

Def: A lattice L with $\hat{0}$ & $\hat{1}$ is complemented if $\forall s \in L \; \exists t \in L$ s.t. $s \wedge t = \hat{0}$ & $s \vee t = \hat{1}$.

If t is unique for each s, we say L is uniquely complemented

If every $[u,v]$ of L is complemented, then L is relatively complemented

Def: An atom of a finite lattice L is any $s \gtrdot \hat{0}$.

- L is atomic (resp point lattice) if every $l \in L$ is a join of atoms ($\hat{0} = \bigvee_{\emptyset}$)

Prop: Fix L a finite upper-semimodular lattice ($s \wedge t \lessdot s, t \Rightarrow s, t \lessdot s \vee t$)

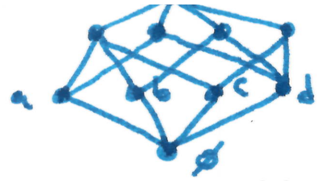
TFAE: (1) L is relatively complemented

(2) L is atomic

Def: A finite upper-semimodular lattice satisfying (1) or (2) is called a finite geometric lattice

Examples: ① Finite point configurations S in affine space V.

$L(S) := \{ S \cap W \mid W \text{ is an affine subspace of } V\}$

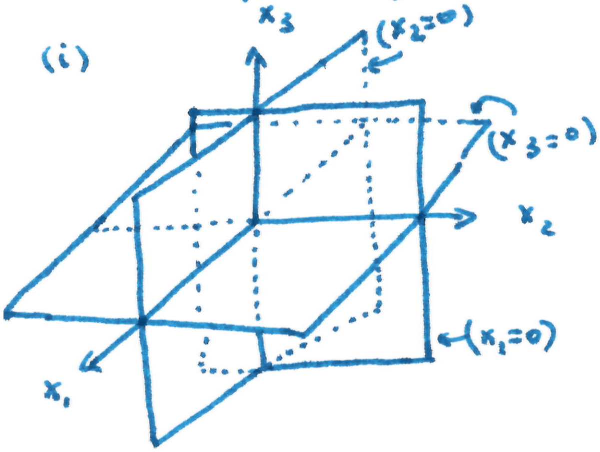
Ex: S  $\subseteq \mathbb{R}^2 = V$ 

$L(S) = \{\emptyset\} \cup \{d, a, b, c\} \cup \{abc\} \cup \{ad\} \cup \{bd\} \cup \{c, d\} \cup V$

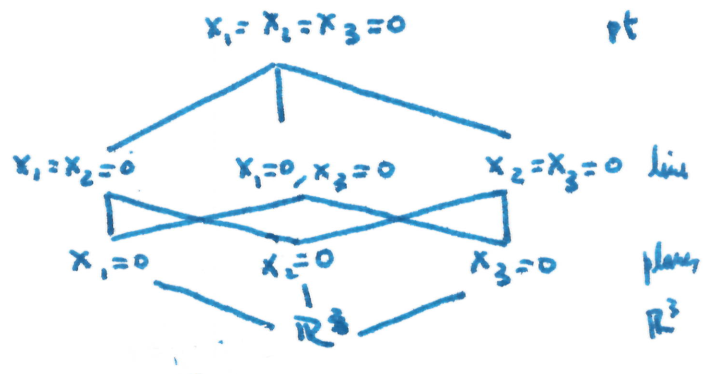
② Hyperplane arrangements in K^n.

$\mathcal{A}$ hyp. arrangement $\Rightarrow \mathcal{L}(A)$ = intersection poset of $\mathcal{A}$, ordered by reverse inclusion

(i)



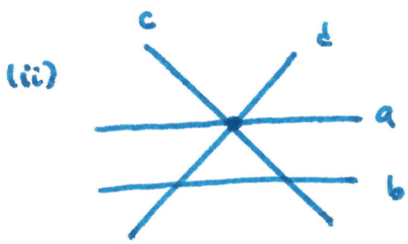
central



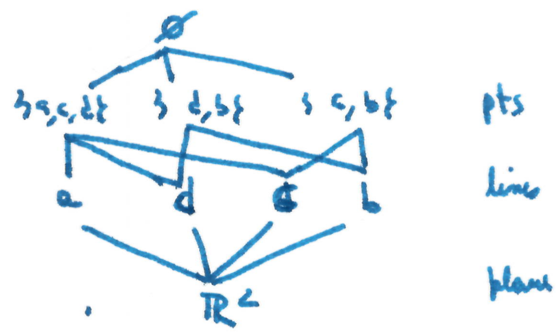
$\mathcal{L}(A)$

$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\}$

(ii)



non-central



$\mathcal{L}(A)$

$\mathcal{A} = \{a, b, c, d\} \subseteq \mathbb{R}^2$

Atoms = hyperplanes