Lecture XXXVII : Incidence algebras

TODAY: Build algebras out of locally finite posets

Recall: K-algebra = K-vector space with multiplication

Ex.: • $\mathbb{C}[x]$ algebra over $\mathbb{C}$

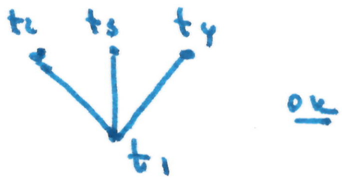
• $U(n,\mathbb{C})$ = upper-triangular matrices over $\mathbb{C}$ ($\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$)

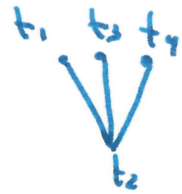
§1 Finite Posets

• Fix P a finite poset with $p = |P|$ elements. & K a field

• Label the elements $t_1, \ldots, t_p$ so that $t_i < t_j$ in $P \Rightarrow i < j$

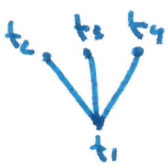
ie label: $P \longrightarrow$ p-chain is order preserving bijection $\rightsquigarrow$ linear extension of P

Ex 

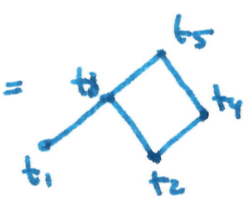
ok , 

not ok.

Def $I(P,K) := \{A \in M(p,K) \mid A_{ij} = 0$ if $t_i \not\leq_P t_j\}$ (rows/cols: labeled by $t_1, \ldots t_p$)

Ex: ① $P =$ 

$\rightsquigarrow$ $A = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \end{array}$ $*$ = entry is ALLOWED to be nonzero.

② $P =$ 

$\rightsquigarrow$ $A = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} \\ \begin{bmatrix} * & 0 & * & 0 & * \\ 0 & * & * & * & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \end{array}$

Obs: $A \in U(P,K)$ by def of labeling! (upper triangular matrices with $*$ in diagonal).

Prop/Def: For a fixed finite poset (with a labeling, ie linear extn.) P, the set $I(P,K)$ is a K-subalgebra of $U(|P|,K)$. It is called the Incidence Algebra of P.

Note: Only hard part is to check $I(P,K)$ is closed under multiplication.

3f/ Pick $A, B \in I(\mathcal{P}, K)$ & take $A \cdot B$. Need to show $t_i \not\leq t_j \Rightarrow (AB)_{ij} = 0$.

But $(A \cdot B)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$

Since $t_i \not\leq t_j \Rightarrow \not\exists\, k$ with $t_i \leq t_k \leq t_j$ (transitivity of $\leq$).

So no matter what k is, either $A_{ik} = 0$ or $B_{kj} = 0$. Thus, $(AB)_{ij} = 0$ $\square$

Obs: $A_{ij} \neq 0 \Rightarrow t_i \leq t_j$

So any $A \in I(\mathcal{P}, K)$ defines a function on closed intervals $[t_i, t_j]$ of $\mathcal{P}$ to K (finite subposets for locally finite $\mathcal{P}$'s).

§2 Locally Finite Posets

Def: Let $\underline{\mathrm{Int}(\mathcal{P})}$ be the set of closed intervals of $\mathcal{P}$ (Each $I \in \mathrm{Int}(\mathcal{P})$ is finite for $\mathcal{P}$ locally finite poset).

Define $I(\mathcal{P}, K) = \{ f : \mathrm{Int}(\mathcal{P}) \longrightarrow K \}$ with operations:

(i) $(f+g)([x,y]) = f([x,y]) + g([x,y]) \quad \forall\, x \leq y$ in $\mathcal{P}$

(ii) $(cf)([x,y]) = c\, f([x,y])$ ——— $\forall\, c \in K$.

(iii) $(f * g)([x,y]) = \sum_{x \leq z \leq y} f([x,z])\, g([z,y])$ ——— [CONVOLUTION]

Multiplicative identity? $\mathbb{1}([x,y]) = \delta_{xy} =: \delta$ (matrices: I_d). $\rightsquigarrow$ $\boxed{\delta * f = f * \delta = f \;\; \forall f \in I(\mathcal{P}, K)}$

Notation: $f(x,y) = f([x,y])$.

Prop: Fix $f \in I(\mathcal{P}, K)$. TFAE:

(1) f has a left inverse (ie $g \in I(\mathcal{P}, K)$: $g * f = \delta$)

(2) f —— right —— (———— $f * g = \delta$)

(3) f has a 2-sided inverse

(4) $f(t,t) \neq 0 \quad \forall\, t \in \mathcal{P}$.

Proof: $f * g = \delta \iff (f*g)(x,y) = \sum_{x \leq z \leq y} f(x,z)\, g(z,y) = \delta_{xy}$

$\iff \forall\, x \in \mathcal{P}$: $f(x,x)\, g(x,x) = 1$ & $\forall\, x < y$ in $\mathcal{P}$. $g(x,y) = -f(x,x)^{-1} \sum_{x \leq z < y} f(x,z)\, g(z,y)$

- So (2) $\Rightarrow f(x,x) \neq 0 \quad \forall x \in \mathcal{P}$ (4)
- Furthermore, from (4) we can reconstruct $g(x,y)$ using
  - $g(x,x) = (f(x,x))^{-1} \in K \quad \forall x \in \mathcal{P}$
  - $g(x,y) = -(f(x,x))^{-1} \sum_{x \lneq z \leq y} f(x,z)\, g(z,y)$ ⇝ $g(y,y)$ gives $g(z,y)$ $z \lessdot y$ ..... get $g(x,y)$.
- $g(x,y)$ only depends on $[x,y]$ & only defined for $x \leq y$ so $g \in I(\mathcal{P}, K)$.

Similarly (1) $\Leftrightarrow$ (4).

Combining them gives (4) $\Leftrightarrow$ (1) & (2) $\equiv$ (3) $\quad [fg = hf = \delta \Rightarrow g = h]$

Note that by construction, if $\mathcal{P}$ is a finite poset with linear order $t_1 < \dots < t_{|\mathcal{P}|}$ then $f : \mathrm{Int}\,\mathcal{P} \longrightarrow K$ ⇝ $[f] \in U_{|\mathcal{P}|}(K)$ upper triangular matrix with $[f]_{ij} = 0$ if $t_i \not\leq t_j$

§3. Main example: Zeta function ζ

Def: Given $\mathcal{P}$ locally finite poset, the zeta function of $\mathcal{P}$ is

$$\zeta(x,y) := 1 \quad \text{for } x \leq y \text{ in } \mathcal{P}.$$

Note: ζ is interesting because it counts things!

Prop 1: $\zeta^{k}(x,y) = \#\{\text{multichains of length } k \text{ from } x \text{ to } y \text{ in } \mathcal{P}\}$ $(x \leq x_1 \leq \dots \leq x_k = y)$

convolution product!

Pf/ $k=1$ ⇝ length 1 chains: $x \leq y$

$k=2$ ⇝ $\zeta^2(x,y) = \sum_{x \leq z \leq y} \zeta(x,z)\zeta(z,y) = \sum_{x \leq z \leq y} 1 = 2\text{-chains}$

By induction: $\zeta^{k+1}(x,y) = (\zeta^k * \zeta)(x,y) = \sum_{x \leq z_{k-1} \leq y} \zeta^k(x, z_{k-1})\, \zeta(z_{k-1}, y)$

$$\underset{(IH)}{=} \sum_{x \leq z_1 \leq \dots \leq z_{k-1} \leq y} \zeta(x,z_1)\,\zeta(z_1,z_2) \cdots \zeta(z_{k-1},y)$$

$$= \sum_{x \leq z_1 \leq \dots \leq z_{k-1} \leq y} 1 = \#\{\text{multichains of length } k \text{ from } x \text{ to } y \text{ in } \mathcal{P}\}$$