Lecture XXXVIII: Zeta & Moebius functions

Recall: P locally finite poset, then $I(P,K) = \{f : \mathrm{Int}(P) \to K\}$ with
$\mathrm{Int}(P) = \{[x,y] : x \le y \text{ in } P\}$

- pointwise $\cdot_K$ & $+$
- convolution product $(f * g)(x,y) = \sum_{x \le z \le y} f(x,z)\, g(z,y)$.

P finite & labeling $t_1, \dots, t_{|P|}$ $(t_i \le_P t_j \Rightarrow i \le j)$ ⇝ upper Δ matrix $[f]_{ij}$ $(= 0 \text{ if } t_i \not\le_P t_j)$

Main example: $\zeta(x,y) = 1$ if $x \le y$ ($* = 1$ in matrix form)

Prop 0: $f \in I(P,K)$ invertible $\iff$ $f(x,x) \neq 0 \ \forall x \in X$.

Prop 1: $\zeta^k(x,y) = \#\{$multichains $\overset{\in \mathbb{Z}}{\text{of}}$ length k from x to y in $P\}$ (INTERPRETATION: = if char $K = 0$ & $\equiv$ mod char K otherwise)

$x = x_0 \le x_1 \le \cdots \le x_{k-1} \le x_k = y$

Proof by $\overset{\in K}{\text{induction}}$ on k. ($k=0$: 0-chains on $[x,y] = \delta(x,y)$)

Prop 2: $(\zeta - \delta)^k(x,y) = \#\{$chains of length k from x to y in $P\} =: C_k(x,y)$

δ " 1 in $I(P,K)$

$x = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = y$

($\equiv$ mod char K if char $K = p > 0$)

Proof: $(\zeta - \delta)(x,y) = \begin{cases} 1 & x < y \\ 0 & x = y \end{cases}$ $\qquad (\zeta-\delta)^0(x,y) \equiv \delta(x,y) = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases}$ counts 0-chains from x to y

By induction on k: $(\zeta - \delta)^k(x,y) = \sum_{x < x_1 < \cdots < x_{k-1} < y} 1 =: C_k(x,y)$ ∎

We would like to have an invertible function in $I(P,K)$:

$$(2\delta - \zeta)(x,y) = \begin{cases} -1 & \text{if } x < y \\ 1 & \text{if } x = y \end{cases} \in I(P,K) \text{ \& invertible.}$$

Prop 3: $(2\delta - \zeta)^{-1}(x,y) = C_0(x,y) + C_1(x,y) + \cdots$ (finite sum because $[x,y]$ is finite)

$= \text{TOTAL } \#$ chains from x to y in P ($\equiv$ mod char K if char $K > 0$)

Proof: If ℓ = length of longest chain in $[x,y]$, we have:

$(\zeta - \delta)^{\ell+1}(x,y) = 0$ by Prop 2 & $C_0(x,y) + C_1(x,y) + \cdots = \sum_{k=0}^{\ell} C_k(x,y)$.

So $\delta(x,y) = [\delta - (\zeta - \delta)^{\ell+1}](x,y) = ([\delta - (\zeta-\delta)] * [\delta + (\zeta-\delta) + (\zeta-\delta)^2 + \cdots + (\zeta-\delta)^\ell])(x,y)$

$\boxed{\delta^2 = \delta \quad (\zeta-\delta)*\delta = \delta * (\zeta - \delta)}$

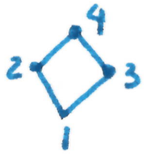
$$= \left((2\delta - \zeta) * (\delta + (\zeta-\delta) + \cdots + (\zeta-\delta)^{\ell}) \right)_{(x,y)}$$

Conclude: $(2\delta - \zeta)^{-1}_{(x,y)} = (\delta + (\zeta-\delta) + (\zeta-\delta)^2 + \cdots + (\zeta-\delta)^{\ell})_{(x,y)}$

$$\underset{\substack{\uparrow \\ \text{Prop 2}}}{=} C_0(x,y) + C_1(x,y) + C_2(x,y) + \cdots + C_{\ell}(x,y). \quad \blacksquare$$

Q: What, if anything, does ζ^{-1} count? (ζ^{-1} exists because $\zeta(x,x) \neq 0 \ \forall x \in P$)

Def: The Moebius function of P is $\mu = \zeta^{-1}$.

Ex: 

$$\zeta = \begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightsquigarrow \mu = \zeta^{-1} = \begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

So $\mu(1,4) = 1$, $\mu(1,3) = -1$.

Claim: $\mu * \zeta = \delta$ is equivalent to recursively defining:

(1) $\mu(x,x) = \zeta^{-1}(x,x) = 1 \quad \forall x \in P$

and (2) $\displaystyle\sum_{x \le z \le y} \mu(x,z) = 0$ for $x<y$ (i.e. $\displaystyle\mu(x,y) = -\sum_{x \le z \lneq y} \mu(x,z)$)

Proof: Def of $*$ & def of ζ & δ.

Thm (Hall's Thm): For P a locally finite poset, we have that:

$$\mu(x,y) = C_0(x,y) - C_1(x,y) + C_2(x,y) - C_3(x,y) + \cdots$$

$$= \sum_{j \ge 0} (-1)^j C_j(x,y)$$ (sum is finite because P is finite / locally finite)

Proof: We use similar technique as before:

$$\mu_{(x,y)} = \zeta^{-1}_{(x,y)} = (\delta + (\zeta - \delta))^{-1}_{(x,y)} \underset{\substack{\uparrow \\ \text{geom series}}}{=} (\delta - (\zeta-\delta) + (\zeta-\delta)^2 - (\zeta-\delta)^3 + \cdots)_{(x,y)}$$

$$= C_{0\,(x,y)} - C_{1\,(x,y)} + C_{2\,(x,y)} - C_{3\,(x,y)} + \cdots$$

Obs: • If $x \neq y$, $C_0(x,y) = 0$ & $C_1(x,y) = 1$

• View $\mu(x,y)$ as an "Euler characteristic" for ~~posets~~ finite & locally finite posets.

Corollary: P loc finite poset: $\displaystyle\mu(x,y) = \sum_{\substack{C \text{ chain} \\ \text{from } x \text{ to } y}} (-1)^{\ell(C)}$ [resembles Sieve Methods]

<u>Theorem</u> (Möbius Inversion): Let P be a finite poset, and let $f,g : P \to K$. The following statements are equivalent:

(1) $$g(x) = \sum_{y \le x} f(y) \qquad \forall x \in P$$

(This is similar to PIE!)

(2) $$f(x) = \sum_{y \le x} g(y)\, \mu(y,x) \qquad \forall x \in P$$

<u>Remark</u>: We can give a second proof of PIE using the <u>dual form</u> of Möbius inversion:

<u>Dual Möbius inversion formula</u>: Let P be a finite poset & fix: $f,g : P \to K$. TFAE

(1) $$g(x) = \sum_{y \ge x} f(y) \qquad \forall x \in P$$

(2) $$f(x) = \sum_{y \ge x} g(y)\, \mu(x,y) \qquad \forall x \in P$$

$(\mu_{P^*}(y,x) = \mu_P(x,y))$

<u>Proof</u> (Möbius Inversion) $K^P = \{f : P \to K\}$ is a K-vector space.

- $I(P,K)$ acts (on the right) on K^P via:

$$(\underset{K^P}{h} \cdot \underset{I(P,K)}{\phi})(x) = \sum_{y \le x} h(y)\, \phi(y,x) \in K^P$$

(i) $h \cdot \delta = h \quad \forall h$ ✓

(ii) $(h \cdot \phi) \cdot \psi = h \cdot (\phi * \psi) \quad \forall h \in K^P,\ \phi, \psi \in I(P,K)$

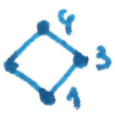
(easy from def)

- Möbius Inversion: $\qquad f \cdot \zeta = g \overset{\cdot \mu}{\underset{\cdot \zeta}{\Longleftrightarrow}} f = g \cdot \mu$.

. $$(f \cdot \zeta)(x) = \sum_{y \le x} f(y)\, \zeta(y,x) = \sum_{y \le x} f(y)$$

. $$(g \cdot \mu)(x) = \sum_{y \le x} g(y)\, \mu(y,x) \qquad \blacksquare$$

<u>Note</u>: In matrix form: f, g are row vectors, ζ & μ are upper $\triangle$ matrices.

<u>Example</u>: $P =$ 

$$f = [f_1, f_2, f_3, f_4]$$

$$g = [f_1, f_1+f_2, f_1+f_3, f_1+f_2+f_3+f_4] = [f_1,f_2,f_3,f_4] \begin{bmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}.$$

$$f = g \cdot \mu = g \cdot \begin{bmatrix} 1&-1&-1&1\\ 0&1&0&-1\\ 0&0&1&-1\\ 0&0&0&1 \end{bmatrix}$$