Lecture XXXIX : Computing Möbius functions

Recall: P finite poset $\quad \zeta(x,y) = 1$ for $x \le y \quad$ ($\zeta : \mathrm{Int}(P) \to \underset{\text{field}}{K}$)

$\underset{\underset{I(P,K)}{\cap}}{\mu = \mu_P} = \zeta^{-1} : \mathrm{Int}(P) \to K \quad$ How? (1) invert matrix

(2) $\begin{cases} \mu(x,x) = 1 & \forall x \in P \\ \mu(x,y) = -\sum_{x \le z \lneq y} \mu(x,z) & \forall x < y. \end{cases}$

Möbius inversion (a dual version) $\quad P$ finite poset, $f, g : P \to K$. TFAE:

(1) $g(x) = \sum_{y \le x} f(y) \quad \forall x \in P$ $\qquad\|\qquad$ (1') $g(x) = \sum_{y \ge x} f(y) \quad \forall x$

(2) $f(x) = \sum_{y \le x} g(y)\, \mu(y,x) \quad \forall x \in P$ $\qquad\|\qquad$ (2') $f(x) = \sum_{y \ge x} g(y)\, \mu(x,y) \quad \forall x$

$\underset{\in K^{|P|}}{[f]} = \underset{\in K^{|P|}}{[g]} \cdot [\mu] \quad \Leftrightarrow \quad \underset{\in K^{|P|}}{[g]} = \underset{\in K^{|P|}}{[f]} \cdot [\zeta]$

Pf/ $I(P,K)$ acts on the right on $\underset{K\text{-v.s.}}{K^P}$ via $(h * \phi)(x) = \sum_{y \le x} h(y)\, \phi(y,x)$. $\square$

Example: Fix n finite sets $S_1, \dots, S_n$

P = poset of intersections among $S_1, \dots, S_n$ ordered by inclusion

($\hat{1} = S_1 \cup \dots \cup S_n$ = empty intersection)

For $T \in P$: $\quad \cdot\ g(T) = |T|$

$\qquad\qquad\ \ \cdot\ f(T) = \#\{t \in T \mid t \text{ is not in any } T' \lneq T,\ T' \in P\}$

Note $f, g \in K^P$. $\quad ; \quad f(\hat{1}) = 0 \quad , \quad$ <u>g satisfies (1) in Möbius inversion</u>

GOAL: Compute $g(\hat{1}) = \sum_{T \le \hat{1}} f(T)$ without explicitly knowing $f(T)$.

Möbius inv gives $0 = f(\hat{1}) = \sum_{T \le \hat{1}} g(T)\, \mu(T, \hat{1}) = g(\hat{1}) \cdot 1 + \sum_{T < \hat{1}} |T|\, \mu(T, \hat{1})$

$$\Rightarrow \boxed{g(\hat{1}) = -\sum_{T < \hat{1}} |T|\, \mu(T, \hat{1}).}$$

Ex: $S_1 = \{1, 2, \underline{5, 6}\}$

$S_2 = \{3, \underline{5, 6}, 7\}$

$S_3 = \{4, \underline{5, 6}, 8\}$

$\rightsquigarrow P = $
$$\begin{tikzcd}[column sep=small]
 & \hat{1} = [8] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
S_1 & S_2 & S_3 \\
 & \{5,6\} = S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = S_1 \cap S_2 \cap S_3 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

$$[\zeta] = \begin{array}{c} \begin{array}{ccccc} S_1 \cap S_2 & S_1 & S_2 & S_3 & \hat{1} \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} S_1 \cap S_2 \\ S_1 \\ S_2 \\ S_3 \\ \hat{1} \end{array} \end{array}$$

$$\left(\sum_{x\le x'\le x''}\mu_P(x,x')\right)\left(\sum_{y\le y'\le y''}\mu_Q(y,y')\right) = \delta_{xx''}\,\delta_{yy''} \overset{?}{=} 0$$

Since $(x,y)<(x'',y'')$ we have either $x=x''$ & $y<y''$
or $x<x''$ & $y\le y''$

In both cases $\delta_{xx'}\,\delta_{yy'}=0$ ✓ $\square$

Alternative proof: $I(P\times Q,\ K)\simeq I(P,K)\otimes_K I(Q,K)$ as vector spaces
and $\zeta_{P\times Q}=\zeta_P\otimes\zeta_Q$.

$n=p_1^{a_1}\cdots p_r^{a_r}$ $p_i\neq p_j$

<u>Corollary</u>: Explicit formulas for $B_n\simeq \underline{2}^n$ & $D_n\simeq(\underline{a_1+1})\times\cdots\times(\underline{a_r+1})$

(1) $\mu_{B_n}(T,S)=(-1)^{|S-T|}$ for $T\le S$ in B_n

(2) $\mu_{D_n}(v,s)=\begin{cases}(-1)^t & \text{if } \frac{s}{v} \text{ is a product of } t \text{ \underline{distinct} primes (exponents}=1)\\ 0 & \text{else}\end{cases}$

Pf/(1) $T=(a_1,\dots,a_n)$ $a_i\le b_i$ 0 or 1. $(a_i=1 \Leftrightarrow i\in T)$
$S=(b_1,\dots,b_n)$ $(b_i=1 \Leftrightarrow i\in S)$

$$\mu_{B_n}(T,S)=\mu_{\underline{2}}(a_1,b_1)\cdots\mu_{\underline{2}}(a_n,b_n)=(-1)^{\#\{i\mid a_i=0\ \&\ b_i=1\}=|S-T|}$$

$\mu_{\underline{2}}(a_1,b_1) =$ 1 if $a_1=b_1$
-1 else $(a_1=0<b_1=1)$

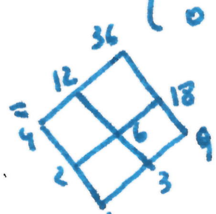
<u>Ex</u>: $\mu(12,1235)$ in B_5 $=\mu_{\underline{2}^5}((11000),(11101))=(-1)^2$

(2) is similar
$s=p_1^{b_1}\cdots p_r^{b_r}$ $c_i\le b_i\le a_i$ $\forall i$
$v=p_1^{c_1}\cdots p_r^{c_r}$

$$\mu_{D_n}(v,s)=\mu_{\underline{a_1+1}}(c_1,b_1)\cdots\mu_{a_r+1}(c_r,b_r)=\begin{cases}0 & \text{if } p_i^2\mid \frac{s}{v} \text{ for some } i\\ (-1)^t & t=\#\{c_i=b_i+1\}.\end{cases}$$

$\mu_{\underline{a_1+1}}(c_1,b_1)=\begin{cases}1 & \text{if } c_1=b_1\\ -1 & \text{if } c_1=b_1+1\\ 0 & \text{else}\end{cases}$

<u>Ex</u>: $D_{36}=\underline{3}\times\underline{3}=$



$\mu_{D_{36}}(2,\underset{=2^2\cdot3^2}{36})=0$ $\frac{36}{2}=\frac{3\cdot3\cdot2}{\text{not distinct}}$

$=\mu_{\underline{3}}(\underset{=(-1)}{1,2})\cdot\mu_{\underline{3}}(\overset{=0}{0,2})$

$$\rightsquigarrow [\mu] = \begin{array}{c} \begin{array}{ccccc} S_1\cap S_2 & S_1 & S_2 & S_3 & \hat{1} \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & \boxed{2} \\ 0 & 1 & 0 & 0 & \boxed{-1} \\ 0 & 0 & 1 & 0 & \boxed{-1} \\ 0 & 0 & 0 & 1 & \boxed{-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} S_1\cap S_2 \\ S_1 \\ S_2 \\ S_3 \\ \hat{1} \end{array}$$

$$|S_1 \cup S_2 \cup S_3| = g(\hat{1}) = -\Big(\underbrace{|S_1\cap S_2|}_{=2}\,2 - \underbrace{|S_1|}_{=4}(-1) - \underbrace{|S_2|}_{=4}(-1) + \underbrace{|S_3|}_{=4}(-1)\Big)$$

$$= 4 + 4 + 4 - 2\cdot 2 = 8 \quad \text{as we wanted!}$$

§2 Techniques of Computation: Chains & Products

Prop 1: C_n chain $1 < 2 < \cdots < n$
$$\mu(i,j) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i+1 = j \\ 0 & \text{else} \end{cases} \quad (\text{indep of } n \,!!)$$

Pf/Check
$$[\zeta * \mu] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & \ddots & & \vdots \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & 0 \\ & 1 & -1 & \\ & & 1 & -1 \\ 0 & & \ddots & \ddots \\ & & & 1 \end{bmatrix} = I_{n\times n}.$$

Consequence: Möbius inversion on C_n states:

$$g(j) = \sum_{i=1}^{j} f(i) \quad \forall j \geq 1 \iff \begin{array}{l} f(1) = g(1) \\ f(2) = -g(1) + g(2) \\ f(3) = -g(2) + g(3) \end{array} \cdots \equiv \boxed{\begin{array}{l} f(1) = g(1) \\ f(j) = g(j) - g(j-1) \\ \quad \forall j \geq 2 \end{array}}$$

$\Delta(g(j))$

$g = [f_1 \cdots f_n] \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \ddots & \vdots \\ & & 1 \end{bmatrix}$ j-col

$f = [g_1 \cdots g_n] \begin{bmatrix} 1 & -1 & 0 & 0 \\ & 1 & \ddots & \\ 0 & & 1 & 0 \\ & & & -1 \\ & & & 1 \end{bmatrix}$ j

Def: Finite difference operator for $g(1), \ldots, g(m), \ldots$ is $\boxed{\Delta(g(j)) := g(j) - g(j-1)}$ $j \geq 2$

Obs: A sequence g obtained from deg k polynomials satisfy $\Delta^{k+1}(g) = 0$ (Δ behaves like $\frac{d}{dx}$)

Thm 1 (Product) Assume P, Q are finite & consider $P\times Q$ finite poset. Then

$$\mu_{P\times Q}((x,y),(x',y')) = \mu_P(x,x')\,\mu_Q(y,y').$$

Pf/ Show (RHS) satisfies recursion defining $\mu_{P\times Q}$. + initial conditions

- Initial conditions: $\mu_P(x,x)\cdot\mu_Q(y,y) = 1\cdot 1 = 1$ ✓
- Recursion: $\displaystyle\sum_{(x,y)\leq(x',y')\leq(x'',y'')} \mu_P(x,x')\,\mu_Q(y,y') \overset{?}{=} 0$ for all $(x,y) \lneq (x'',y'')$

$\mu_{D_{36}}(3, \underset{2\cdot 3^2}{18}) = \mu_{\underline{3}}(0,1) \cdot \mu_{\underline{3}}(1,2) = (-1)^2 = 1$ & $\frac{18}{3} = 2\cdot 3$ prod of 2 distinct primes

$\mu_{D_{36}}(\hat{0}, \hat{1}) = \mu_{\underline{3}}(0,2)\ \mu_{\underline{3}}(0,2) = 0 \leadsto \hat{1} \neq 2 \vee 3$ (atoms of D_{36})

<u>Application 1</u>: Möbius inversion for D_N:

$$g(n) = \sum_{d \mid n} f(d) \quad \text{for all } n \mid N \iff f(n) = \sum_{d\mid n} g(d)\, \underline{\mu(d,n)} \ \forall\, n \mid N.$$

<u>Application 2</u>: Dual Möbius inversion for $B_n \Rightarrow$ PIE ($g = N_{=}$ & $f = N_{\geq}$)

Exact properties satisfied; at least satisfied

(1) $N_{\geq}(T) = \sum_{Y \supseteq T} N_{=}(Y)$ (TRUE) & (2) $N_{=}(T) = \sum_{Y \supseteq T} N_{\geq}(Y)\ \underbrace{\mu_{B_n}(T,Y)}_{= (-1)^{|Y-T|}}$ (PIE)

((1) $\Rightarrow$ (2) from Möbius inv.)

§ Two more techniques: Weisner's & Cross-cut Thms

<u>Weisner's Thm</u>: Fix L a finite lattice & $a \in L$ with $a \neq \hat{1}$. Then

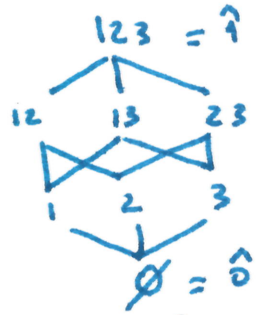
$$\sum_{t:\, t\wedge a = \hat{0}} \mu(t, \hat{1}) = 0 \qquad \& \qquad \sum_{t:\, t \vee a = \hat{1}} \mu(\hat{0}, t) = 1$$

(Typical applications: a = atom or coatom).

<u>Crosscut Thm</u> (Special case) Fix L a finite lattice & assume $\hat{0}$ is <u>NOT</u> a meet of <u>coatoms</u> (ie $x \lessdot \hat{1}$) Then, $\mu(\hat{0}, \hat{1}) = 0$

Similarly, if $\hat{1}$ is not a join of <u>atoms</u> (ie $x \gtrdot \hat{0}$), then $\mu(\hat{0}, \hat{1}) = 0$.

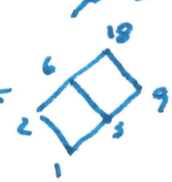
<u>Ex</u>: For B_3:



$a = \{1\}$ in Weisner's Thm

$0 = \mu(\emptyset, \hat{1}) + \mu(2, \hat{1}) + \mu(3, \hat{1}) + \mu(23, \hat{1})$
$= (-1)^3 + (-1)^2 + (-1)^2 + (-1)^1$
$= 0$ ✓

For $D_{\underset{2\cdot 3^2}{18}} = \underline{2} \times \underline{3} =$



2 & 3 are atoms but $6 = 2 \vee 3 \neq 18$.

So crosscut gives $\mu(\hat{0}, \hat{1}) = \mu(1, 18) = 0$

<u>Check</u>: $\mu(1,18) = \mu_{\underline{2}}(0,1)\ \mu_{\underline{3}}(0,2) = (-1)\cdot(0) = 0$ ✓