Lecture XL: Combinatorics of Hyperplane arrangements I

§1 Basic definitions:

Fix K field (typically $K=\mathbb{R}$ or $\mathbb{C}$, but could also have char $K>0$) & $V \cong K^n$

Def A. **hyperplane** H in V is an affine linear subspace of codim $= 1$ ($\dim_{\text{aff}} H = n-1$)

An (affine) arrangement $\mathcal{A}$ is a finite collection of hyperplanes in V.

Def: We say $\mathcal{A}$ is <u>centerless</u> if its <u>center</u> $C(\mathcal{A}) = \bigcap_{H\in\mathcal{A}} H$ is $\emptyset$.

If $0 \in C(\mathcal{A})$, then $\mathcal{A}$ is <u>central</u>.

- We identify each H in $\mathcal{A}$ with $\{f_H, \alpha_H\} \in V^* \times K$

  Equation for H: $f_H(v) = \langle \underset{V^*}{\underset{\cap}{f_H}}, \underset{V}{\underset{\cap}{v}} \rangle = \alpha_H$ ($\langle , \rangle$ pairing between V^* & V)

  View f_H as generator of $(H-p)^\perp$ with $p \in H$.

- If $L|K$ is a field extension, we can consider the L-extended arrangement $\mathcal{A}_L$ of $\mathcal{A}$ in $V_L = V_K \otimes_K L$ with the same defining equations of $\mathcal{A}$.

  <u>Typical</u>: $K=\mathbb{R}$ & $L=\mathbb{C}$

Questions: Combinatorics of $\mathcal{A}$, Topology of $V \setminus \mathcal{A} \subseteq V$, nice compactifications?

§2 GADGET 1: Intersection Poset: Say $|\mathcal{A}| = n$

Def: $L(\mathcal{A}) = \{$ non-empty intersections $\bigcap_{i\in I} H_i : I \subseteq [n]\}$ where $V = \bigcap_{i\in\emptyset} H_i$

- $L(\mathcal{A})$ is a <u>poset</u> under <u>reverse inclusion</u> $s \leq t \iff s \supseteq t$.

  <u>Name</u>: Intersection poset.

- $\hat{0} = V \in L(\mathcal{A})$, $\exists\ \hat{1} \in L(\mathcal{A}) \iff \mathcal{A}$ is central

- <u>Names</u>: $s \in L(\mathcal{A})$ is also called a <u>flat</u> of the arrangement $\mathcal{A}$

- Atoms of $L(\mathcal{A})$ = hyperplanes of $\mathcal{A}$ (atoms $= \{s \mid s > \hat{0}\}$)

- $L(\mathcal{A})$ is graded

Lemma: $L(\mathcal{A})$ is a <u>meet semi-lattice</u> with

$$X, Y \in L(\mathcal{A}) : X \wedge Y := \bigcap_{X\cup Y \subseteq Z \in L(\mathcal{A})} Z$$

(Index set is non-empty because $V \in L(\mathcal{A})$)

⚠ $X \vee Y = X \cap Y$ does not work only because the intersection could be $\emptyset$!

Corollary: ① If $\mathcal{A}$ is central, then $L(\mathcal{A})$ is a lattice. Furthermore, it is a geometric lattice (upper semi-modular + any $s \in L(\mathcal{A})$ is a join of atoms). In particular

(*). $L(\mathcal{A})$ is graded with $\rho : L(\mathcal{A}) \longrightarrow \mathbb{Z}$

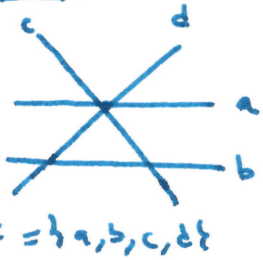
$$X \longmapsto \operatorname{codim}(X) = n - \dim X$$

· $\rho(X \wedge Y) + \rho(X \vee Y) \leq \rho(X) + \rho(Y)$ $\qquad$ $(\rho(X \wedge Y) \leq \rho(X+Y))$

. Any flat is a join of atoms (Tautological!)

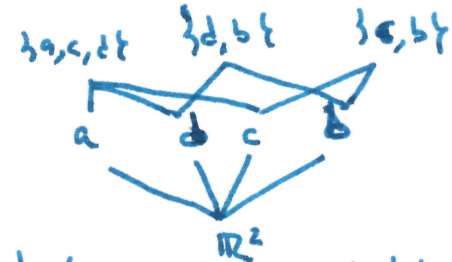
② For any $\mathcal{A}$, every interval of $L(\mathcal{A})$ is a geometric lattice.

Examples:

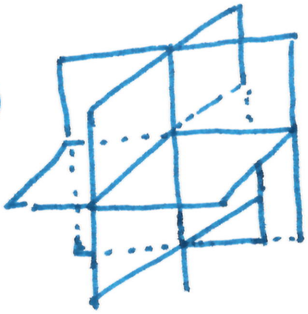
① 

$\mathcal{A} = \{a, b, c, d\}$

non-central

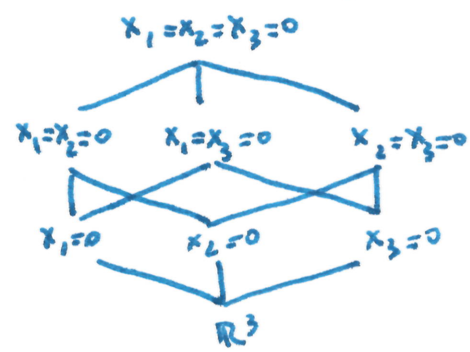
$L(\mathcal{A}) =$ 

not a lattice! (missing $\hat{1}$). → (add it to get a lattice)

② 

central

$\mathcal{A} = B_3$ = Boolean arrangement

$L(\mathcal{A}) =$ 

$rk = 3 = n$ (amb dim)

$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\}$

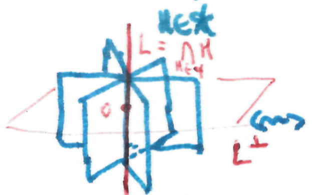
$$\begin{array}{cccl} \updownarrow & \updownarrow & \updownarrow & \text{"normal"} \\ e_1 & e_2 & e_3 & \text{vector (in } V^*) \text{ configuration} \end{array}$$

In general, if $\hat{1} \neq \{p\}$, then the $rk\, L(\mathcal{A}) = \operatorname{codim}\left(\bigcap_{H \in \mathcal{A}} H\right) \neq n$.

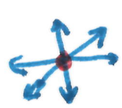
This is equiv to $\operatorname{span}\langle f_H : H \in \mathcal{A} \rangle \subseteq V^*$ having $\dim < n$.

We can take $V/\bigcap_{H \in \mathcal{A}} H \simeq W \simeq \left(\bigcap_{H \in \mathcal{A}} H\right)^{\perp}$ if char $k = 0$, & reduce to the $\hat{1} = \{p\}$ case (example of an essential arrangement)

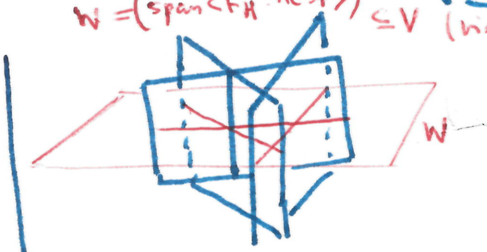
$W = (\operatorname{span}\langle f_H : H \in \mathcal{A} \rangle) \subseteq V$ (via $V^* \simeq V$)

Ej: 

in $\mathbb{R}^3$



essential in $\mathbb{R}^2$



$\mathcal{A}_W = \{H \cap W \mid H \in \mathcal{A}\}$ same information as $\mathcal{A}$ & it's an "essential arrangement"

Essentialization of $\mathcal{A}$

$ess(\mathcal{A})$

Def: $\operatorname{Rank}(\mathcal{A}) = \dim(\operatorname{span}\langle f_H : H \in \mathcal{A} \rangle) \subseteq V^*$

$\mathcal{A}$ is an essential arrangement if: $\operatorname{Rank}(\mathcal{A}) = n$

In char $\mathbb{K} > 0$, we should be a bit careful ($W \cap W^{\perp} = \{0\}$ may fail!)

Why? $L(\mathcal{A}) \simeq L(\mathrm{ess}(\mathcal{A}))$ & $\mathrm{ess}(\mathcal{A})$ is essential.

<u>Examples</u>: ① <u>Boolean</u> arrangement B_n in K^n

$$B_n = \bigcup_{i=1}^{n} \{x_i = 0\}$$

<u>Claim</u>: $L(B_n) \simeq B_n$ Boolean lattice

$$X = H_{i_1} \cap \dots \cap H_{i_j} \longmapsto \{i_1, \dots, i_j\}$$

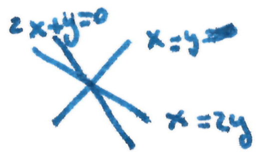
② <u>Braid</u> arrangement Br_n in K^n

$$Br_n = \bigcup_{1 \leq i < j \leq n} (x_i - x_j = 0)$$

Not essential $C(Br_n) = K \cdot \underline{1}$ $(\mathrm{rank}(Br_n) = n-1)$

$\rightsquigarrow$ View in $W = \{x \in K^n \mid x_1 + \dots + x_n = 0\}$

<u>Eg</u>: Br_3 :



$\subseteq (x + y + z = 0)$

<u>Claim</u>: $L(Br_n) \simeq \Pi_n$ partition lattice

$$X \longmapsto (A_1, \dots, A_k)$$
$$i, j \in A_k \Leftrightarrow x_i = x_j \quad \forall x \in X$$

§3 GADGET 2: Möbius functions on $L(\mathcal{A})$

<u>Recall</u>: $\begin{cases} \mu(x,x) = 1 & \forall x \in L(\mathcal{A}) \\ \sum\limits_{X \leq Z \leq Y} \mu(X, Z) = 0 & \text{for } X \lneq Y \text{ in } L(\mathcal{A}) \end{cases}$

<u>Ex</u>: $\mu_{B_n}(x, y) = (-1)^{|y-x|}$ when we view $X \in B_n$, $Y \in B_n$ with $X \leq Y$.

We use μ to define 2 <u>numerical combinatorial invariants</u> for $\mathcal{A}$:

<u>Def 1</u>: $\chi(\mathcal{A}, t) = \sum\limits_{X \in L(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim X}$ (characteristic polynomial)

<u>Def 2</u>: $\pi(\mathcal{A}, t) = \sum\limits_{X \in L(\mathcal{A})} \mu(\hat{0}, X)(-t)^{\mathrm{codim} X} = t^n \chi(\mathcal{A}, -t^{-1})$ (Poincaré polynomial)

Why invariants? If $\mathcal{A}$ & $\mathcal{B}$ are combinatorially equivalent arrangements (meaning $L(\mathcal{A}) \simeq L(\mathcal{B})$)

then $\chi(\mathcal{A}, t) = \chi(\mathcal{B}, t)$

$\pi(\mathcal{A}, t) = \pi(\mathcal{B}, t)$

So we can distinguish arrangements if these polynomials are different.

For this, we need formulas for computing them (next time!)