Lecture XLI: Combinatorics of hyperplane arrangements II

Recall: Fix $\mathcal{A} \subseteq K^n$ hyperplane arrangement $\rightsquigarrow L(\mathcal{A})$ intersection poset w/ inverse incl. ($\rho: L(A) \to \mathbb{Z}_{\geq 0}$, $\rho(X) = \operatorname{codim} X$)
$\operatorname{Rank}(\mathcal{A}) = \dim(\operatorname{span}\langle f_H : H \in \mathcal{A}\rangle \subseteq (K^n)^*)$. (Lattice if $\mathcal{A}$ is central) . intervals are geometric lattices

Two main numerical combinatorial invariants:

(1) Characteristic Polynomials $\quad \chi(\mathcal{A}, t) = \sum\limits_{X \in L(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim X} = \underset{(X=\emptyset)}{t^n} - \underset{\rho(X)=1}{|\mathcal{A}|\, t^{n-1}} + \cdots$

(2) Poincaré Polynomials $\quad \pi(\mathcal{A}, t) = \sum\limits_{X \in L(\mathcal{A})} \mu(\hat{0}, X)\, (-t)^{\operatorname{codim} X} = t^n \chi(\mathcal{A}, -\frac{1}{t})$

Why? $K = \mathbb{C}$: $\pi(\mathcal{A}, t) =$ Poincaré polynomial of the cohomology ring of $M_{\mathcal{A}} = \mathbb{C}^n \setminus \bigcup\limits_{H \in \mathcal{A}} H$.
$= \sum\limits_{k \geq 0} \operatorname{rank} H^k(M_A, \mathbb{Z})\, q^k$. [Orlik-Solomon, 1980]

Ex ① $B_n = \bigcup\limits_{i=1}^{n} \{x_i = 0\}$ Boolean arrangement $\quad L(B_n) \cong B_n$ Boolean lattice.

- $\chi(\mathcal{A}, t) = \sum\limits_{X \in B_n} \underbrace{\mu(\emptyset, X)}_{(-1)^{|X|}}\, t^{n-|X|} = \sum\limits_{X \in B_n} (-1)^{|X|}\, t^{n-|X|} = (t-1)^n$. $\quad \operatorname{codim} X = \rho(X) = |X|$; alt. signs
- $\pi(\mathcal{A}, t) = \sum\limits_{X \in B_n} (-1)^{|X|} (-t)^{n-|X|} = \sum\limits_{X \in B_n} t^{n-|X|} = (t+1)^{|X|}$; only positive!

Thm: If $X \leq Y$ in $L(\mathcal{A})$, then $\mu(X, Y)$ has sign $= (-1)^{\rho(Y) - \rho(X)}$

(True for any finite upper semimodular lattice, like $[\hat{0}, Y]$ in $L(\mathcal{A})$) (alternates with $l([X, Y])$.)

Consequence: Coefficients of $\pi(\mathcal{A}, t)$ are all positive (they count things!)

[J. Huh 2010] Coefficients of $\pi(\mathcal{A}, t)$ are log-concave $\rightsquigarrow$ $\boxed{\text{matroids \& characteristic polynomials, Hodge Theory}}$

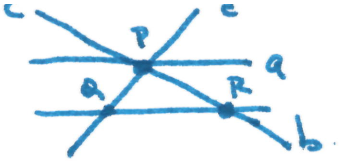
② $Br_n = \bigcup\limits_{1 \leq i < j \leq n} \{x_i - x_j = 0\}$ $\quad L(Br_n) \cong \Pi_n$ partition lattice $\quad (\rho(\pi) = n - \#\pi$ [number of blocks])

$$\chi(\mathcal{A}, h) = \sum_{\pi \in \Pi_n} \mu(1|2|\cdots|n, \pi)\, t^{n - \operatorname{codim} = \rho(\pi)} \underset{\substack{\uparrow \\ \text{[Möbius inversion]}}}{=} t(t-1)\cdots(t-n+1)$$

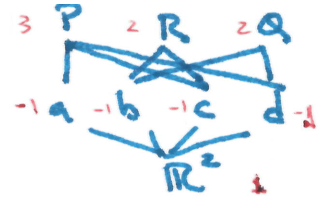
Applications to Topology:

Thm: $\mathcal{A} \subseteq \mathbb{R}^n$. [Zaslavsky 1975]
(1) $\#\{\text{regions of } M_{\mathcal{A}} = \mathbb{R}^n \setminus \bigcup\limits_{H \in \mathcal{A}} H\} = (-1)^n \chi_{\mathcal{A}}(-1)$
(2) $\#\{$____________, relatively bounded$\} = (-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1)$.

Eg:



$L(\mathcal{A}) =$



write $\mu(\hat{0}, X)$ next to each X

$\mu(\mathbb{R}^2, P) = -\mu(\mathbb{R}^2, a) - \mu(\mathbb{R}^2, b) - \mu(\mathbb{R}^2, c) - \mu(\mathbb{R}^2,\mathbb{R}^2) = 3 - 1 = 2$

$\mu(\mathbb{R}^2, R) = -(-2+1) = 1$

$\mu(\mathbb{R}^2, Q) = 1$

$\Rightarrow \chi(\mathcal{A}, t) = t^2 - 4t + (1+1+2)t^0 = t^2 - 4t + 4$

9 regions
1 bounded
$\chi_{\mathcal{A}}(-1) = 9$
$(-1)^2 \chi_{\mathcal{A}}(1) = 1 \quad rk(\mathcal{A}) = 2$

Q: How to compute $\chi(\mathcal{A}, t)$?

(1) Whitney's Thm: $$\chi(\mathcal{A}, t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central} \\ (C(\mathcal{B}) \neq \emptyset)}} (-1)^{|\mathcal{B}|}\, t^{n - \mathrm{Rank}(\mathcal{B})}$$ for $\mathcal{A} \subseteq K^n$ arrangement

$\mathcal{B} \to X \in L(\mathcal{A})$ is not a 1-1 correspondence! (see last page)

$\mathrm{Rank}(\mathcal{B}) = \mathrm{codim}(C(\mathcal{B}))$

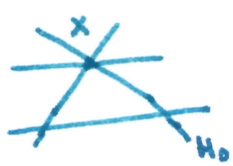
$|\mathcal{B}| = \#$ hyperplanes in $\mathcal{B}$

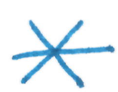
(2) Recurrence:

Fix $H_0 \in \mathcal{A}$ (= distinguished hyperplane).

Def:
- $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ ($X \geq H$) subarrangement ($\leftrightarrow [\hat{0}, X]$ in $L(\mathcal{A})$)
- $\mathcal{A}^X = \{H \cap X \mid X \in \mathcal{A},\ X \not\subseteq H,\ X \cap H \neq \emptyset\}$ arrangement in X
- $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ (deleted arrangement wrt H_0)
- $\mathcal{A}'' = \mathcal{A}^{H_0}$ (restricted ——— in H_0; contraction of $\mathcal{A}$ to H_0)
- $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ distinguished triple wrt H_0 ⇝ useful for induction!

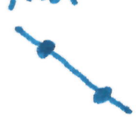
Eg:

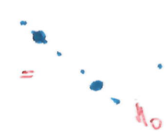


$\mathcal{A}_X =$ 

$\mathcal{A}^X = \emptyset$

$\mathcal{A}_{H_0} = \{H_0\}$

$\mathcal{A}^{H_0} =$  $= 2$ pts in H^0

$\mathcal{A}' =$  , $\mathcal{A}'' =$ 

Obs: $L(\mathcal{A}_X) \cong [\hat{0}, X]$ in $L(\mathcal{A})$

$L(\mathcal{A}^X) \cong \{S \in L(\mathcal{A}) \mid S \geq X\}$

Deletion - Restriction Thm: For $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ distinguished triples wrt H_0, we have

$$\chi(\mathcal{A},t) = \chi(\mathcal{A}',t) - \chi(\mathcal{A}'',t)$$

$$\pi(\mathcal{A},t) = \pi(\mathcal{A}',t) + t\pi(\mathcal{A}'',t)$$

Pf/ Enough to show 1st identity. Use Whitney's Thm:

$$\chi(\mathcal{A},t) = \underbrace{\sum_{\substack{H_0\notin B\\ B\subset \mathcal{A}\\ \text{central}}} (-1)^{|B|}\, t^{n-\text{rank}(B)}}_{=\chi(\mathcal{A}',t)} + \underbrace{\sum_{\substack{H_0\in B\\ B\subset\mathcal{A}\\ \text{central}}} (-1)^{|B|}\, t^{n-\text{rank}(B)}}_{\overset{?}{=}\, -\chi(\mathcal{A}'',t)}$$

We identify $H_0\in B\subset \mathcal{A}$ central with $B'' = B^{H_0} \subset K^{n-1}$ $\quad$ B'' central in $\mathcal{A}^{H_0}$ "$\mathcal{A}''$", $|B''| = |B|-1$

$\text{rank}(B'') = \text{rank}(B) - 1$.

$B \to B''$ not 1-1 !!

$$\underbrace{-\sum_{\substack{B''\subseteq \mathcal{A}^{H_0}\\ \text{central}}} (-1)^{|B''|}\, t^{n-1-\text{rank}(B'')}}_{=-\chi(\mathcal{A}'',t)} \overset{\triangle}{\underset{(*)}{=}} -\sum_{\substack{H_0\in B\subseteq\mathcal{A}\\ \text{central}}} (-1)^{|B|-1}\, t^{n-\text{rank}(B)} = \sum_{\substack{H_0\in B\\ B\subset\mathcal{A}\\ \text{central}}} (-1)^{|B|}\, t^{n-\text{rank} B} \quad \blacksquare$$

To "fix" this, we need the following Lemma:

Lemma: B'' central subarrangement of $\mathcal{A}''$, then:

$$\sum_{\substack{H_0\in B\\ B\subseteq \mathcal{A}\\ \text{central}\\ B^{H_0}=B''}} (-1)^{|B|} = (-1)^{|B''|+1}$$

[Lemma 2.2 in Dimca's book]

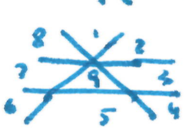
In (*):
$$\sum_{\substack{H\in B\\ B\subset A\\ \text{central}}} (-1)^{|B|} x^{n-\text{rank}(B)} = \sum_{\substack{B''\subset\mathcal{A}''\\ \text{central}}} \sum_{\substack{H_0\in B\\ B\subseteq A\\ \text{central}\\ B^{H_0}=B''}} (-1)^{|B|}\, x^{\overset{n-1-\text{rank}(B'')}{\overset{=}{n-\text{rank}(B)}}} = \sum_{\substack{B''\subset\mathcal{A}''\\ \text{central}}} (-1)^{|B''|+1}\, x^{n-1-\text{rank}(B'')}$$

$$= -\chi(\mathcal{A}'',t) \qquad \square$$

§2 Counting regions for $\mathbb{R}$-arrangements:

Def: A *region* of $\mathcal{A}\subseteq\mathbb{R}^n$ is a connected component of the complement $M_{\mathcal{A}}$.

Write $\mathcal{R}(\mathcal{A}) = \{\text{regions of } \mathcal{A}\}$ & $r(\mathcal{A}) = \#\mathcal{R}(\mathcal{A})$, $bb(\mathcal{A}) = \#\{X\in\mathcal{R}(\mathcal{A}) \text{ bounded}\}$

Eg: 

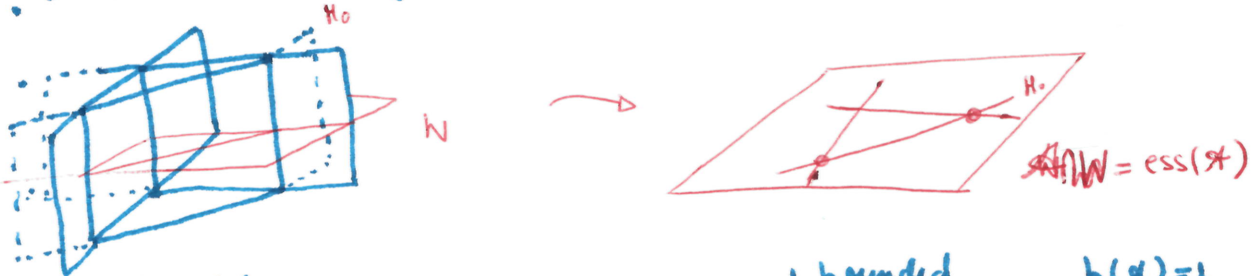
$r(\mathcal{A}) = 9$, 1 bounded.

Def (Relatively bounded) Write $W = \operatorname{span}\{H^{\perp} : H \in \mathcal{A}\} \subseteq \mathbb{R}^n$

- Every $R \in \mathcal{R}(\mathcal{A})$ open & convex so $\simeq$ int $(B \subseteq \mathbb{R}^n)$
- $\mathcal{R}(\mathcal{A}) \longrightarrow \mathcal{R}(ess(\mathcal{A}))$ is a bijection

$$R \longmapsto R \cap W$$

$R \in \mathcal{R}(\mathcal{A})$ is relatively bounded if $R \cap W$ is bounded.

- For essential arrangements, $W = \mathbb{R}^n$ so bounded = rel. bounded
- 

no bounded
7 unbounded.

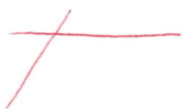
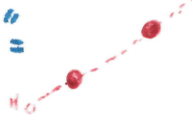
1 bounded $\quad b(\mathcal{A}) = 1$
6 unbounded $\quad r(\mathcal{A}) = 7$

Lemma: $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ triple with distinguished hyperplane H_0. Then:

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$$

$$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}'') & \text{if } rank(\mathcal{A}) = rank(\mathcal{A}') \quad (= 1 + rank\,\mathcal{A}'') \\ 0 & \text{if } rank(\mathcal{A}) = rank(\mathcal{A}') + 1 \end{cases}$$

Pf idea: look at regions of $\mathcal{A}'$ cut out by H_0: regions of $\mathcal{A}'$ will be cut into 2 regions in $\mathcal{A}$. Identify them with regions in $\mathcal{A}''$ ("the cut")

Ex: $\mathcal{A}' =$  $\mathcal{A}'' =$ 

$$\begin{cases} 7 = 4 + 3 \\ 1 = 0 + 1 \end{cases}$$

$rank(\mathcal{A}) = 2 = rank(\mathcal{A}')$

$b(\mathcal{A}') = 0 \qquad b(\mathcal{A}'') = 1$

$r(\mathcal{A}') = 4 \qquad r(\mathcal{A}'') = 3$

$rank(\mathcal{A}') = 2 \qquad rank(\mathcal{A}'') = 1$

Thm [Zaslavsky 1975]

$$r(\mathcal{A}) = (-1)^n \chi(\mathcal{A}, -1)$$
$$b(\mathcal{A}) = (-1)^{rank(\mathcal{A})} \chi(\mathcal{A}, 1)$$

Pf/ · True for $\mathcal{A} = \emptyset$
· Same recurrence is satisfied (Lemma + deletion-restriction)
· Argument for b is more delicate b/c we have 2 cases.

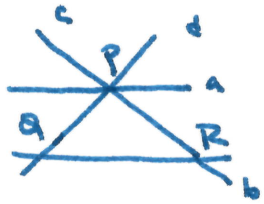
$d(\mathcal{A}) = (-1)^{rank(\mathcal{A})} \chi(\mathcal{A}, 1)$

(1) If $rank(\mathcal{A}) = rank(\mathcal{A}') \Rightarrow d(\mathcal{A}) = d(\mathcal{A}') + d(\mathcal{A}'')$
$b(\mathcal{A}) = b(\mathcal{A}') + b(\mathcal{A}'')$

(2) If $rank(\mathcal{A}) = rank(\mathcal{A}') + 1$, then $b(\mathcal{A}) = 0$ & $L(\mathcal{A}') \simeq L(\mathcal{A}'')$ & $d(\mathcal{A}) = 0$ by Deletion-Restr.

Ex(2) 

Example for Whitney's Thm: We record all central subarrangements & their corresponding centers.



B	$C(B)$	rank(B)	$\lvert B\rvert$
$\emptyset$	$\mathbb{R}^2$	0	0
$\{a\}$	$\{a\}$	1	1
$\{b\}$	$\{b\}$	1	1
$\{c\}$	$\{c\}$	1	1
$\{d\}$	$\{d\}$	1	1
$\{ac\}$	P	2	2
$\{ad\}$	P	2	2
$\{cd\}$	P	2	2
$\{cb\}$	R	2	2
$\{db\}$	Q	2	2
$\{acd\}$	P	2	3

$$\Rightarrow \chi(\mathcal{A},t) = x^2-4x+(5-1) = x^2-4x+4$$

Examples for region count:

regions traversed by H_0 in $\mathcal{A}$ $\overset{1\text{-to-}1}{\longleftrightarrow}$ regions in $\mathcal{A}''$

① 

	$\mathcal{A}$	$\mathcal{A}'$	$\mathcal{A}''$
$r =$	9	6	3
$b =$	1	0	1

② 

	$\mathcal{A}$	$\mathcal{A}'$	$(\mathcal{A}')^{ess}$	$\mathcal{A}''$
$r =$	6	3	$\boxed{3}$	3
$b =$	0	0?	$\boxed{1}$	1

$$\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}')+1$$
$$\Downarrow$$
$$b(\mathcal{A}) = 0.$$