MATH 7141 - Algebraic Geometry I

Homework 1

Zariski topology, basic properties of affine varieties and defining ideals

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume $\mathbb{K}$ is an arbitrary field.

Definition: A topological space X is called **Noetherian** if there is no infinite strictly decreasing chain $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$ of closed subsets of X.

Problem 1. Show that any affine variety (with the Zariski topology) is a Noetherian topological space. In particular, show that the affine space $\mathbb{A}^n_{\mathbb{K}}$ is Noetherian.

Problem 2. Show that every Noetherian topological space is compact (i.e., every open cover of such a space has a finite subcover). Conclude from Problem 1 that every open subset of an affine variety is compact in the Zariski topology.

Problem 3. Let X be an irreducible affine subvariety of $\mathbb{A}^n_{\mathbb{K}}$ and U be any non-empty open subset of X (with respect to the Zariski topology). Show that U is dense in X. Show that the irreducibility of X is necessary by providing a counter-example to this density statement.

Problem 4. Show that a subvariety V of $\mathbb{A}^n_{\mathbb{K}}$ is irreducible if, and only if, for every variety $W \subsetneq V$, the difference $V \smallsetminus W$ is Zariski dense in V.

Problem 5. Find the irreducible components of the affine variety $V(x - yz, xz - y^2) \subset \mathbb{A}^3_{\mathbb{C}}$.

Problem 6. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X} \subset \mathbb{A}^n$.

Recall that for two ideals $\mathfrak{a}_1, \mathfrak{a}_2$ of a ring R the **ideal quotient** (or **colon ideal**) equals

$$(\mathfrak{a}_1 : \mathfrak{a}_2) = \{f \in R : f\mathfrak{a}_2 \subset \mathfrak{a}_1\}.$$

Problem 7. Let X_1 and X_2 be affine subvarieties of $\mathbb{A}^n_{\mathbb{K}}$. Show that

$$I(\overline{X_1 \smallsetminus X_2}) = (I(X_1) : I(X_2)).$$

Problem 8. Consider two ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathbb{K}[x_1, \ldots, x_n]$. Show that $V(\mathfrak{a}_1) = V(\mathfrak{a}_1 + \mathfrak{a}_2) \cup V((\mathfrak{a}_1 : \mathfrak{a}_2))$.

Problem 9. Consider two ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathbb{K}[x_1, \ldots, x_n]$.

(i) Prove that $\overline{V(\mathfrak{a}_1) \smallsetminus V(\mathfrak{a}_2)} \subseteq V((\mathfrak{a}_1 : \mathfrak{a}_2))$.

(ii) Show that the inclusion in the previous item can be strict by considering the ideals $\mathfrak{a}_1 = (x^2(y-1))$ and $\mathfrak{a}_2 = (y)$ of $\mathbb{C}[x, y]$.

Problem 10. Consider two affine subvarieties V, W of $\mathbb{A}^n_{\mathbb{K}}$. Show that $V = (V \cap W) \cup \overline{V \smallsetminus W}$.

Problem 11. Let $X \subset \mathbb{A}^n_{\mathbb{K}}$ and $Y \subset \mathbb{A}^m_{\mathbb{K}}$ be two affine varieties.

(i) Prove that their product $X \times Y \subset \mathbb{A}^{n+m}_{\mathbb{K}}$ is also an affine variety.

(ii) Show that $X \times Y$ is irreducible if both X and Y are irreducible.

Problem 12. Consider the following parameterization of a subset X of $\mathbb{A}^2_{\mathbb{C}}$, via $x = t/(1+t)$, and $y = 1 - 1/t^2$ for $t \neq -1, 0$ in $\mathbb{C}$.

(i) Determine $I(\overline{X})$.

(ii) Compute $\overline{X} \smallsetminus X$.

Problem 13.(The Twisted Cubic) Assume $\mathbb{K}$ is infinite and consider $X = V(y - x^2, z - x^3) \subseteq \mathbb{A}^3_{\mathbb{K}}$. This set is a curve in $\mathbb{A}^3$, known as the twisted cubic.

(i) Show that $I(X) = (y - x^2, z - x^3) \subset \mathbb{K}[x, y, z]$.

(ii) Show that $y^2 - xz \in I(X)$ by writing it explicitly as a polynomial combination of the generators of $I(X)$.

Problem 14. Assume $\mathbb{K}$ is infinite, and consider the subset X of $\mathbb{A}^3_{\mathbb{K}}$ parameterized by $t \mapsto (t, t^3, t^4)$.

(i) Show that X is an affine subvariety of $\mathbb{A}^3_{\mathbb{K}}$.

(ii) Determine generators for $I(X)$.