MATH 7141 - Algebraic Geometry I

Homework 2

Primary decompositions

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume R is a commutative Noetherian ring and $\mathbb{K}$ is an arbitrary field. In addition, $r(\cdot)$ always denotes the radical of an ideal.

Problem 1. Show that an ideal $\mathfrak{q} \subset R$ is primary if, and only if, every zero divisor in $R/\mathfrak{q}$ is nilpotent.

Problem 2. Let $\mathfrak{m} \subset R$ be a maximal ideal. Prove that $\mathfrak{m}^n$ is primary for every $n \geq 1$.

Problem 3. Let $\mathfrak{q} \subset R$ be an ideal. If $r(\mathfrak{q})$ is maximal, then show that $\mathfrak{q}$ is primary.

Problem 4. Assume R is Noetherian and let $\mathfrak{p} \subset R$ be a prime ideal. Prove that $R_{\mathfrak{p}}$ is Artinian if, and only if, $\mathfrak{p}$ is a minimal prime ideal of R.

Problem 5. Let $\mathfrak{a} \subset R$ be an ideal. Prove that there exists $n \geq 1$ such that $r(\mathfrak{a})^n \subseteq \mathfrak{a}$.

Problem 6. Prove the following equality is true in the polynomial ring $\mathbb{K}[x, y]$:

$$(x^2, y) \cap (x, y^2) = (x, y)^2.$$

In addition, show that the three ideals above are primary. (*Hence, primary does not imply irreducible.*)

Problem 7. Prove that $(4, t) \subset \mathbb{Z}[t]$ is a primary ideal. Verify that $r((4, t)) = (2, t)$ which is a maximal ideal in $\mathbb{Z}[t]$. Prove that $(2, t)^2 \subsetneq (4, t) \subsetneq (2, t)$. (*Hence, a primary ideal need not be power of a prime.*)

Problem 8. Consider a primary ideal $\mathfrak{q}$ in R with radical $r(\mathfrak{q}) = \mathfrak{p}$. Let $S \subsetneq R$ be a multiplicatively closed set.

(i) Prove that $S \cap \mathfrak{q} \neq \emptyset$ if, and only if, $S \cap \mathfrak{p} \neq \emptyset$.
(ii) Assume that $S \cap \mathfrak{p} = \emptyset$. Show that $S^{-1}\mathfrak{q}$ is a primary ideal in $S^{-1}R$ and $r(S^{-1}\mathfrak{q}) = S^{-1}\mathfrak{p}$ in $S^{-1}R$.

Problem 9. Let $R = \mathbb{K}[x, y]$, $I = (x^2, xy) \subset R$. Take $\mathfrak{p} = (x)$ and $\mathfrak{q}_n = (x^2, xy, y^n)$ for each $n \geq 2$. Prove that

(i) $\mathfrak{p}$ is a prime ideal;
(ii) each $\mathfrak{q}_n$ is primary and $r(\mathfrak{q}_n) = (x, y)$;
(iii) $\mathfrak{p} \cap \mathfrak{q}_n = I$.

Hence, we have infinitely many distinct minimal primary decompositions. Notice that they all have the same set of associate primes $\{(x), (x, y)\}$.

Definition: We say $\mathfrak{a} \subset \mathbb{K}[x_1, \ldots, x_n]$ is a **monomial ideal** if it is generated by monomials in the variables $x_1, \ldots, x_n$.

Problem 10. Characterize monomial ideals in $\mathbb{K}[x_1, \ldots, x_n]$ that are:

(i) prime;
(ii) irreducible;
(iii) radical;
(iv) primary.

Problem 11. Provide an explicit algorithm for computing the radical of a monomial ideal and use it to determine $r((xy^3z, x^2z)) \subset \mathbb{K}[x, y, z]$

Problem 12. (Bonus) Provide an explicit algorithm for computing a primary decomposition of a monomial ideal and use it to decompose the ideal $(xy^3z, x^2z) \subset \mathbb{K}[x, y, z]$.