MATH 7141 - Algebraic Geometry I
Homework 3

Saturation ideals, Hilbert Nullstellensatz and Noether Normalization

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume $\mathbb{K}$ is an arbitrary field, and $r(\cdot)$ denotes the radical of an ideal.

Definition: Given two ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathbb{K}[x_1, \ldots, x_n]$, we define the **saturation ideal** of $\mathfrak{a}_1$ with respect to $\mathfrak{a}_2$ as:

$$(\mathfrak{a}_1 : \mathfrak{a}_2^\infty) := \{f \in \mathbb{K}[x_1, \ldots, x_n] : \text{ for all } g \in \mathfrak{a}_2, \text{ there exists } N \geq 0 \text{ with } f\, g^N \in \mathfrak{a}_1\}.$$

Problem 1. Consider two $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathbb{K}[x_1, \ldots, x_n]$.

(i) Prove that $(\mathfrak{a}_1 : \mathfrak{a}_2^\infty)$ is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$.

(ii) Show that the following inclusions hold: $\mathfrak{a}_1 \subseteq (\mathfrak{a}_1 : \mathfrak{a}_2) \subseteq (\mathfrak{a}_1 : \mathfrak{a}_2^\infty)$;

(iii) Show that given $N \geq$ we have that $(\mathfrak{a}_1 : \mathfrak{a}_2^\infty) = (\mathfrak{a}_1 : \mathfrak{a}_2^N)$ if, and only if $(\mathfrak{a}_1 : \mathfrak{a}_2^{N+1}) \subseteq (\mathfrak{a}_1 : \mathfrak{a}_2^N)$;

(iv) Conclude that for some $N \gg 1$ we have $(\mathfrak{a}_1 : \mathfrak{a}_2^N) = (\mathfrak{a}_1 : \mathfrak{a}_2^\infty)$;

(v) Prove that $r((\mathfrak{a}_1 : \mathfrak{a}_2^\infty)) = (r(\mathfrak{a}_1) : \mathfrak{a}_2)$.

(vi) Show that $\mathfrak{a}_2 \subseteq r(\mathfrak{a}_1)$ if, and only if, $(\mathfrak{a}_1 : \mathfrak{a}_2^\infty) = \mathbb{K}[x_1, \ldots, x_n]$.

Problem 2. Consider a collection of ideals $I, J_1, \ldots, J_r$ of $\mathbb{K}[x_1, \ldots, x_n]$. Show that the following identities hold:

$$(I : \sum_{k=1}^{r} J_k) = \bigcap_{k=1}^{r} (I : J_k) \qquad \text{and} \qquad (I : (\sum_{k=1}^{r} J_k)^\infty) = \bigcap_{k=1}^{r} (I : J_k^\infty).$$

Consider two ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathbb{K}[x_1, \ldots, x_n]$. Problem 9 of Homework 1 showed that

$$\overline{V(\mathfrak{a}_1) \smallsetminus V(\mathfrak{a}_2)} \subseteq V((\mathfrak{a}_1 : \mathfrak{a}_2))$$

and gave an example where the inclusion is strict. In that example, the ideals were not radical. The next problem confirms that the reverse inclusion holds when the ideals are radical and the field $\mathbb{K}$ is algebraically closed.

Problem 3. Let $\mathfrak{a}_1, \mathfrak{a}_2$ be two ideals of $\mathbb{K}[x_1, \ldots, x_n]$.

(i) Show that if $\mathbb{K}$ is algebraically closed, then $V(\mathfrak{a}_1 : \mathfrak{a}_2^\infty) \subseteq \overline{V(\mathfrak{a}_1) \smallsetminus V(\mathfrak{a}_2)}$.

(ii) Conclude that if both $\mathfrak{a}_1, \mathfrak{a}_2$ are radical ideals, then $V(\mathfrak{a}_1 : \mathfrak{a}_2) = \overline{V(\mathfrak{a}_1) \setminus V(\mathfrak{a}_2)}$. (*Hint:* Use Problem 9 of Homework 1 and Problem 1 above).

Problem 4. Let $\mathfrak{a} := (x^2 + y^2 - 1, y - 1) \subset \mathbb{K}[x, y]$. Find $f \in I(V(\mathfrak{a}))$ with $f \notin \mathfrak{a}$.

Problem 5. Let $\mathfrak{a} = (f_1, \ldots, f_s) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Then, $f \in r(I)$ if, and only if, 1 belongs to the ideal $\tilde{\mathfrak{a}} = (f_1, \ldots, f_s, 1 - y\, f) \subseteq \mathbb{K}[x_1, \ldots, x_n, y]$.

Problem 6. Let $\mathfrak{a}_1, \mathfrak{a}_2$ be ideals in $\mathbb{K}[x_1, \ldots, x_n]$ Show that their intersection can be computed by consider the extended ideal $(y\, \mathfrak{a}_1 + (1 - y)\, \mathfrak{a}_2) \subset \mathbb{K}[x_1, \ldots, x_n, y]$ and the induced elimination ideal

$$\mathfrak{a}_1 \cap \mathfrak{a}_2 = (y\, \mathfrak{a}_1 + (1 - y)\, \mathfrak{a}_2) \cap \mathbb{K}[x_1, \ldots, x_n].$$

Problem 7. Recall from Problem 13 Homework 1 that $V(y - x^2, z - x^3)$ is the twisted cubic in $\mathbb{A}^3_{\mathbb{R}}$. Show that $V((y - x^2)^2 + (z - x^3)^2) \subseteq \mathbb{A}^3_{\mathbb{R}}$ is also the twisted cubic.

The next exercise generalizes the previous example: it shows that if $\mathbb{K}$ is a field that is not algebraically closed, then any variety $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ can be defined by a single equation.

Problem 8. Consider a field $\mathbb{K}$ which is not algebraically closed.

(i) Show that for each positive integer n, there exists a polynomial $f_n \in \mathbb{K}[x_1, \ldots, x_n]$ whose only solution in $\mathbb{A}^n_{\mathbb{K}}$ is the origin. (*Hint:* Use induction on n. For the case $n = 2$, consider the homogeneization of a polynomial $g \in \mathbb{K}[x_1]$ with no solutions on $\mathbb{K}$.)

(ii) If $W = V(g_1, \ldots, g_s)$ is an affine variety in $\mathbb{A}^m_{\mathbb{K}}$, show that W can be defined by a single equation.

Problem 9. Let $\mathbb{K}$ be an arbitrary field and let S be the subset of all polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ that have no zeros in $\mathbb{A}^n_{\mathbb{K}}$. If $\mathfrak{a}$ is any ideal in $\mathbb{K}[x_1, \ldots, x_n]$ with $\mathfrak{a} \cap S = \emptyset$, show that $V(\mathfrak{a}) \neq \emptyset$. (*Hint:* When $\mathbb{K}$ is not algebraically closed, use Problem 8 above.)

Problem 10. Consider the finitely generated $\mathbb{K}$-algebra $A = \mathbb{K}[x, y]/(y^2 - x^3 + x)$. Determine $m \geq 1$ and elements $z_1, \ldots, z_m \in A$ algebraically independent over $\mathbb{K}$ so that A becomes a finite $\mathbb{K}[z_1, \ldots, z_m]$-algebra.

Problem 11. Consider the finitely generated $\mathbb{K}$-algebra $A = \mathbb{K}[x, y, z]/(xy)$. Determine $m \geq 1$ and elements $z_1, \ldots, z_m \in A$ algebraically independent over $\mathbb{K}$ so that A becomes a finite $\mathbb{K}[z_1, \ldots, z_m]$-algebra.