MATH 7141 - Algebraic Geometry I

Homework 4

Polynomial, rational and regular morphisms; Sheaf theory

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume $\mathbb{K}$ is an arbitrary field.

Problem 1. (Basic Zariski open sets are affine varieties) Given an affine variety $W = V(S) \subseteq \mathbb{A}^n_{\mathbb{K}}$ and $f \in \mathbb{K}[W]$, we consider the basic Zariski open $D_W(f)$ of W. Show that $D_W(f)$ can be viewed as an affine variety in $\mathbb{A}^{n+1}_{\mathbb{K}}$ via $D_W(f) = V(S \cup \{1 - x_{n+1}f\})$. (In particular, it is independent of the choice of representative for $f \in \mathbb{K}[W]$).

Problem 2. Let $\mathbb{K}$ be an algebraically closed field. Show that the ring of regular functions on $U := \mathbb{A}^2_{\mathbb{K}} \smallsetminus \{(0,0)\}$ agrees with $\mathbb{K}[x,y] = \mathbb{K}[\mathbb{A}^2_{\mathbb{K}}]$. Conclude that every regular function on U can be extended to a regular function on $\mathbb{A}^2_{\mathbb{K}}$. Provide a counter-example to this statement if $\mathbb{K}$ is not algebraically closed.

We can generalize the previous problem even further:

Problem 3. Fix an affine variety X and an irreducible affine variety $Y \subseteq X$ defined over an algebraically closed field $\mathbb{K}$. Set $U := X \smallsetminus Y$. If $\mathbb{K}[X]$ is a unique factorization domain, show that $\mathcal{O}_X(U) = \mathbb{K}[X]$ if Y is not defined by the vanishing of a single element of $\mathbb{K}[X]$.

Problem 4. Show that the statement of Problem 3 can fail if $\mathbb{K}[X]$ is not a unique factorization domain.

The following problem provides an alternative definition of rational functions.

Problem 5. Fix two irreducible affine varieties $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ and $W \subseteq \mathbb{A}^m_{\mathbb{K}}$. Consider the collection

$$\mathcal{S} := \{(U, \gamma) : U \subset V \text{ is a non-empty open of } V, \gamma \colon U \to W \text{ is a regular map}\}.$$

We define an equivalence relation $(U, \gamma) \sim (U', \gamma')$ if, and only if, we have the identity of functions $\gamma_{|_{U \cap U'}} \equiv \gamma'_{|_{U \cap U'}}$ on $U \cap U'$. Show that $\mathcal{S}/\sim$ can be identified with the set of rational maps from V to W.

Problem 6. Let $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ be an irreducible affine variety. Show that restrictions maps induce the following injective natural maps of rings for each $p \in W$:

$$\mathcal{O}(W) \to \mathcal{O}_p \to \mathbb{K}(W).$$

Problem 7. Fix an irreducible affine variety $W \subseteq \mathbb{A}^n_{\mathbb{K}}$, and assume $\mathbb{K}$ is algebraically closed. Prove that for any point p of W, the sheaf of regular functions on W at p is isomorphic to the localization of $\mathbb{K}[W]$ at the maximal ideal $\mathfrak{m}_p$ associated to p.

Problem 8. Let $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ be an affine variety and assume $\mathbb{K}$ is algebraically closed. Show that for every point $p \in W$ we have:

$$\mathcal{O}_{W,p} \simeq \mathcal{O}_{\mathbb{A}^n_{\mathbb{K}},p}/I(W)\mathcal{O}_{\mathbb{A}^n_{\mathbb{K}},p}.$$

Here, $I(W)\mathcal{O}_{\mathbb{A}^n_{\mathbb{K}},p}$ denotes the ideal of $\mathcal{O}_{\mathbb{A}^n_{\mathbb{K}},p}$ generated by $\{f/1 : f \in I(W)\}$.

Problem 9. Let $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ and $W \subseteq \mathbb{A}^m_{\mathbb{K}}$ be affine varieties, and consider a morphism $\varphi\colon V \to W$. Show that φ is dominant if, and only if the kernel of the corresponding map $\varphi^*\colon \mathbb{K}[W] \to \mathbb{K}[V]$ on coordinate rings is contained in the nilradical of $\mathbb{K}[W]$.

Problem 10. Let $\mathcal{F}$ be a sheaf on a topological space X, and let $p \in X$. Show that the stalk $\mathcal{F}_p$ is a local object in the following sense: if $U \subseteq X$ is an open neighborhood of p, then $\mathcal{F}_p$ is isomorphic to the stalk at p of the sheaf $\mathcal{F}_{|U}$ on any open set U of X containing p.

Fix a topological space X and a presheaf $\mathcal{F}$ on it. We can build a space $|\mathcal{F}|$ using the stalks of $\mathcal{F}$ and a natural map $p\colon |\mathcal{F}| \to X$ called the projection map, as follows:

$$|\mathcal{F}| = \bigsqcup_{x \in X} \mathcal{F}_x \quad \text{and} \quad p\colon |\mathcal{F}| \to X \quad \text{with } \eta \in \mathcal{F}_x \mapsto x.$$

Problem 11. (A topological space constructed from a presheaf) For each $U \subset X$ open and each $f \in \mathcal{F}(U)$ we define the set

$$\mathcal{N}(U, f) = \{\rho_x(f) \in \mathcal{F}_x : x \in U\}.$$

(i) Show that $\mathcal{B} := \{\mathcal{N}(U, f) : U \subset X \text{ open }, f \in \mathcal{F}(U)\}$ is a basis for a topology on $|\mathcal{F}|$.
(ii) Show that when $|\mathcal{F}|$ is endowed with this topology, the projection map $p\colon |\mathcal{F}| \to X$ is a local homeomeorphism.

Definition: We say a presheaf $\mathcal{F}$ on a topological space X *satisfies the Identity Theorem* if the following condition holds: "For every $U \subseteq X$ open and connected, we have that whenever $f, g \in \mathcal{F}(U)$ satisfy $\rho_p(f) = \rho_p(g)$ for some $p \in U$, then $f = g$."

The theory of complex functions of one variable shows that the Identity Theorems holds for both the sheaf of meromorphic and holomorphic functions. It fails for the sheaf of continuous $\mathbb{C}$-valued functions on a topological space X.

Problem 12. Assume that X is a Hausdorff and locally connected topological space and let $\mathcal{F}$ be a presheaf on X satisfying the Identity Theorem. Then, the topological space $|\mathcal{F}|$ is Hausdorff. (*Hint:* You need to treat two cases when aiming to separate points by opens on $|\mathcal{F}|$; one when the points of $|\mathcal{F}|$ belong to the same stalk, and one where they do not.)

Definition: Consider a topological space X and a presheaf $\mathcal{F}$ on X. Given an open $U \subseteq X$, a *section to* the surjective map $p : |\mathcal{F}| \to X$ over U is given by a map $s \colon U \to |\mathcal{F}|$ with $p \circ s = \mathrm{inc}_U \colon U \hookrightarrow X$.

Typically, we refer to elements $s \in \mathcal{F}(U)$ as "sections" of $\mathcal{F}$ on U. The next problem justifies the use of this terminology.

Problem 13.("Section" terminology for sheaves) Fix X a topological space and a presheaf of sets $\mathcal{F}$ on X. For each open $U \subset X$, consider the set

$$\mathcal{G}(U) := \{s \colon U \to |\mathcal{F}| \text{ continuous sections to } p \text{ over } U\}$$

Consider the map $\phi \colon \mathcal{F}(U) \to \mathcal{G}(U)$ defined by $\phi(f) = (s \colon x \mapsto \rho_x(f) \text{ for all } x \in U)$.

(i) Show that $\mathcal{G}$ defines a sheaf on X with the usual restriction.
(ii) Show that ϕ is well-defined, i.e. $\phi(f) \in \mathcal{G}(U)$ for each $f \in \mathcal{F}(U)$.
(iii) Show that ϕ is a map of presheaves and that the induced map on stalks is a bijection.
(iv) Confirm that ϕ is a bijection when $\mathcal{F}$ is a sheaf. This justifies the terminology "sections" for elements in $\mathcal{F}(U)$.

Problem 14. Fix a locally connected topological space X and an abelian group A. We define the *constant sheaf* $\mathcal{A}$ on X determined by A as follows: for each open U of X, $\mathcal{A}(U)$ is the group of all continuous maps from U to A, where A is endowed with the discrete topology.

(i) Show that $\mathcal{A}$ is a sheaf.
(ii) Furthermore, confirm that $\mathcal{A}$ is the sheafification of the presheaf of abelian groups with assignment $U \mapsto A$ for all $U \neq \emptyset$, whereas $\emptyset \mapsto \{0\}$.

Problem 15. Let $\mathcal{C}$ be the category of Abelian groups or $\mathbb{K}$-vector spaces. Show that a sequence of sheaves

$$\dots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \longrightarrow \dots$$

on a topological space X is exact if and only if for each $p \in X$ the corresponding sequence of stalks is exact.

Problem 16. Let X be a topological space and $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X.

(i) Show that φ is surjective if, and only if, the following condition holds: for every open set $U \subseteq X$, and every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}_{i \in I}$ of U and elements $t_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(t_i) = s_i$ for all $i \in I$.
(ii) Show that if φ is a surjective morphism of presheaves, the maps φ_U need not be surjective.

Problem 17. (Direct sum of sheaves) Consider two sheaves $\mathcal{F}$ and $\mathcal{G}$ of abelian groups or $\mathbb{K}$-vector spaces on a topological space X.

(i) Show that the presheaf $\mathcal{F} \oplus \mathcal{G}$ defined by $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ for each open $U \subseteq X$ is a sheaf.

(ii) Show that $\mathcal{F} \oplus \mathcal{G}$ plays the role of direct sum and direct product in the category where $\mathcal{F}$ and $\mathcal{G}$ are defined.

Problem 18. (Direct limits of sheaves) Consider a direct system of shaves $\{\mathcal{F}_i\}_{i \in I}$ and morphisms of abelian groups or $\mathbb{K}$-vector spaces on a topological space X. We define the *direct limit* of the system to be the sheaf $\varinjlim \mathcal{F}_i$ associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ for each open $U \subseteq X$.

(i) Show that this is a direct limit in the category of sheaves on X, i.e., that it satisfies the following universal property: given a sheaf $\mathcal{G}$ and a collection of morphisms $\mathcal{F}_i \to \mathcal{G}$ compatible with the maps of the direct system, there exists a unique morphism of sheaves $\varinjlim \mathcal{F}_i \to \mathcal{G}$ such that for each i, the original map $\mathcal{F}_i \to \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \to \mathcal{G}$.

(ii) Show that if X is a Noetherian topological space (such as, e.g., an affine variety), then the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular, show that $\varinjlim \mathcal{F}_i(U) = (\varinjlim \mathcal{F}_i)(U)$.

Problem 19. (Inverse limits of sheaves) Consider an inverse system of shaves $\{\mathcal{F}_i\}_{i \in I}$ and morphisms of abelian groups or $\mathbb{K}$-vector spaces on a topological space X. Consider the presheaf $\varprojlim \mathcal{F}_i$ defined by $U \mapsto \varprojlim \mathcal{F}_i(U)$ for each open U of X.

(i) Show that $\varprojlim \mathcal{F}_i$ is a sheaf. It is called the *inverse limit* of the system $\{\mathcal{F}_i\}_{i \in I}$.

(ii) Show that $\varprojlim \mathcal{F}_i$ satisfies the universal property of an inverse limit, i.e., given a sheaf $\mathcal{G}$ and a collection of morphisms $\mathcal{G} \to \mathcal{F}_i$ compatible with the maps of the inverse system, there exists a unique morphism of sheaves $\mathcal{G} \to \varprojlim \mathcal{F}_i$ such that for each i, the original map $\mathcal{G} \to \mathcal{F}_i$ is obtained by composing the maps $\mathcal{G} \to \varprojlim \mathcal{F}_i \to \mathcal{F}_i$.

Problem 20.(Glueing Sheaves) Let X be a topological spaces and $\{U_i\}_{i \in I}$ be an open cover of X. Suppose that we are given for each $i \in I$ a sheaf $\mathcal{F}_i$ on U_i, together with isomorphisms $\varphi_{ij} \colon \mathcal{F}_{i|_{U_i \cap U_j}} \to \mathcal{F}_{j|_{U_i \cap U_j}}$ satisfying:

(1) for each i, $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$, and

(2) for each $i, j, k \in I$ we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Show that there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i \colon \mathcal{F}_{|U_i} \to \mathcal{F}_i$ such that for each $i, j, k \in I$ we have $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. The sheaf $\mathcal{F}$ is called the *glueing* of the sheaves $\mathcal{F}_i$ via the isomorphisms φ_{ij}.