MATH 7141 - Algebraic Geometry I

Homework 5

Projective varieties; rational and regular maps on projective varieties

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume $\mathbb{K}$ is an **infinite** field. Given $n \geq 1$ and $k \in \{0, \ldots, n\}$, the set $U_k \subset \mathbb{P}^n$ denotes the k^{th} standard affine patch on $\mathbb{P}^n$.

Problem 1. Fix a projective variety $V \subseteq \mathbb{P}^n$. Prove that V is irreducible if, and only if, the affine varieties $V \cap U_i \subset U_i \simeq \mathbb{A}^n$ are irreducible for all $i = 0, \ldots, n$. Use this to show that $\mathbb{P}^n$ is irreducible.

Problem 2. Consider the twisted cubic in $\mathbb{A}^2$ defined in Problem 13 of Homework 2.

(i) Compute its projective closure in $\mathbb{P}^2$.
(ii) Compute its homogeneous coordinate ring.

Problem 3. Let $V \subseteq \mathbb{P}^n$ be a non-empty projective variety with $V = V_{\text{proj}}(I)$ for a homogeneous ideal I in $\mathbb{K}[x_0, \ldots, x_n]$. Show that the affine cone $C(V)$ equals $V(I) \subseteq \mathbb{A}^{n+1}_{\mathbb{K}}$.

Problem 4. For each of the following affine varieties (viewed in the affine patch U_0), compute its projective closure and identify the points at infinity that were added:

(i) $W = V(y^2 - x^3 - a\,x - b) \subset \mathbb{K}^2$ where $a, b \in \mathbb{K}$.
(ii) $W = V(x_3^2 - x_1^2 - x_2^2) \subset \mathbb{K}^3$.

Problem 5. Compute the projective closure of $W = V(x_1x_3 - x_2^2, x_1^2 - x_2) \subset \mathbb{A}^3_{\mathbb{K}} \simeq U_0 \subset \mathbb{P}^3$ and find the irreducible decomposition of $\overline{W} \subset \mathbb{P}^3$.

Problem 6. (Dual projective spaces). Let $V \subset \mathbb{P}^n$ be a hyperplane section of $\mathbb{P}^n$, defined by a polynomial $H_{\underline{a}}(\underline{x}) := a_0\,x_0 + a_1\,x_1 + \ldots + a_n\,x_n = 0$. Define

$$\varphi\colon \{\text{hyperplane sections of } \mathbb{P}^n\} \to \mathbb{P}^n \quad V_{\text{proj}}(H_{\underline{a}}(\underline{x})) \mapsto [\underline{a}] \in \mathbb{P}^n.$$

The image of this map is called the *dual projective space* and it is denoted by $(\mathbb{P}^n)^\vee$. Describe the subset of $(\mathbb{P}^n)^\vee$ corresponding to the hyperplanes containing $p = [1 : 0 : \ldots : 0]$.

Definition: Two varieties V and V' in $\mathbb{P}^n$ are *projectively equivalent* if there is an automorphism $A \in \mathbb{P}\,\mathrm{GL}_{n+1}(\mathbb{K})$ of $\mathbb{P}^n$ carrying V onto V'.

Problem 7. Show that two varieties V and V' in $\mathbb{P}^n$ are projectively equivalent if and only if the homogeneous coordinate rings $S(V)$ and $S(V')$ are isomorphic as graded $\mathbb{K}$-algebras.

Problem 8. Consider the rational normal curve C in $\mathbb{P}^3_{\mathbb{K}}$ obtained as the image of the degree three Veronese embedding of $\mathbb{P}^1$.

(i) Show that C is the intersection of the following three quadrics $Q_1 = V_{\mathrm{proj}}(xz - y^2)$, $Q_2 = V_{\mathrm{proj}}(xt - yz)$, $Q_3 = V_{\mathrm{proj}}(yt - z^2)$ in $\mathbb{P}^3$. Furthermore, prove that the intersection of any two of these quadrics strictly contains the rational normal curve.
(ii) Can you determine the irreducible decomposition of these pairwise intersections?

Problem 9. Consider a linear hyperplane $H \subseteq \mathbb{P}^n$ (i.e $H = V_{\mathrm{proj}}(F(\underline{x}))$ where $F(\underline{x})$ is a homogeneous polynomial of degree one in $\mathbb{K}[x_0, \ldots, x_n]$). Show that $\mathbb{P}^n \smallsetminus H$ is an affine variety isomorphic to $\mathbb{A}^n$.

The next exercise provides a coordinate-free version of the d^{th} Veronese embedding.

Problem 10. Let V an $n+1$-dimensional vector space and fix $d \geq 1$. Consider the (non-linear) inclusion $\iota\colon V \hookrightarrow \mathrm{Sym}^d(V)$ where $\iota(v) = v^{\otimes d}$.

(i) Show that this map induces an injective map $\nu_d\colon \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d(V))$.
(ii) Show that by picking appropriate bases of V and $\mathrm{Sym}^d(V)$ the map ν_d agrees with the d-Veronese embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$.

The next exercise provides a coordinate free version of the Segre embedding.

Problem 11. Let V and W be vector spaces over $\mathbb{K}$ of dimensions $m+1$ and $n+1$, respectively. Consider the bilinear map $\varphi\colon V \times W \to V \otimes W$ defined by $(v, w) \mapsto$v⊗w.

(i) Show that φ induces a map $\mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$. Show that the image consists of all rank-1 tensors on $V \times W$.
(ii) Show that by picking appropriate bases on V, W and $V \otimes W$ this map yields the Segre map $\sigma_{m,n}\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$.

Definition: A *degree d hypersurface* in $\mathbb{P}^n$ is the projective vanishing locus of a single homogeneous polynomial of degree d in $\mathbb{K}[x_0, \ldots, x_n]$.

Problem 12. Let X be a degree d hypersurface in $\mathbb{P}^n$. Show that $U := \mathbb{P}^n \smallsetminus X$ admits a non-constant regular morphism to an affine space. (*Hint:* View X as a linear hyperplane section of the image of the d^{th} Veronese map $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ and use Problem 11.)

Definition: A projective variety is *rational* if it is birational to $\mathbb{P}^n$ for some $n \geq 1$.

Problem 13. Show that $\mathbb{P}^n \times \mathbb{P}^m$ is a rational projective variety by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are rational projective varieties, then so is $X \times Y$. (*Hint:* Construct the maps on an affine patch and homogenize accordingly.)

Problem 14. Consider a quadratic form $Q(x, y, z, w)$ in $\mathbb{C}^4$ associated to a rank four symmetric matrix A, i.e. $Q = [x, y, z, w]\, A\, [x, y, z, w]^t$. Let $X = V_{\text{proj}}(Q) \subseteq \mathbb{P}^3_{\mathbb{C}}$.

(i) Show that X is projectively equivalent to $V_{\text{proj}}(xy - zw)$.
(ii) Use the previous item and the Segre embedding to conclude that any quadric of maximal rank in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

The goal of the next two exercises is to show that all quadrics in $\mathbb{P}^n_{\mathbb{C}}$ of maximal rank are rational.

Problem 15. Consider the quadratic hypersurface $X' := V_{\text{proj}}(x_0x_1 - x_2^2 - \ldots - x_n^2) \subseteq \mathbb{P}^n$. Let $p = [1 : 0 : \ldots : 0] \in X'$ and set $H_0 = V_{\text{proj}}(x_0) \subseteq \mathbb{P}^n$.

(i) Show that for any $q \in X'$, the line L passing through p and q intersects H_0 at a single point.
(ii) Use the previous item to define a rational map $\varphi\colon X' \dashrightarrow H_0$, well-defined outside p.
(iii) Show that φ is a birational isomorphism by explicitly constructing the inverse function.

Problem 16. Fix a rank $n + 1$ quadratic form Q in $\mathbb{C}^{n+1}$ and let $X := V_{\text{proj}}(Q) \subseteq \mathbb{P}^n$.

(i) Show that X is projectively equivalent to the quadratic variety $X' \subseteq \mathbb{P}^n$ defined in Problem 15.
(ii) Use Problem 15 to conclude that X is birational to $\mathbb{P}^{n-1}$.

Problem 17. Compute the intersection of the Segre and the Veronese varieties in $\mathbb{P}^5$.

Problem 18. Show that the diagonal $\Delta_{\mathbb{P}^n \times \mathbb{P}^n} = \{(x, y) \in \mathbb{P}^n \times \mathbb{P}^n \colon x = y \text{ in } \mathbb{P}^n\}$ is a projective variety by realizing it as the vanishing loci on some $\mathbb{P}^N$ of an explicit collection of homogeneous polynomials.

Problem 19. Consider a cubic curve X in $\mathbb{P}^2_{\mathbb{C}}$ given by the vanishing loci of a homogeneous degree three polynomial in three variables. Consider the subset U of $X \times X$ defined as:

$$U := \{(a, b) \in X \times X \colon a \neq b, \text{ the line } L_{ab} = \overline{ab} \text{ in } \mathbb{P}^2 \text{ meets } X \text{ at 3 distinct points}\}$$

. We define a map $\varphi\colon U \to X$ via $\varphi(a, b) = p$, where $p = (X \cap L_{ab}) \smallsetminus \{a, b\}$.

(i) Show that $U \subset X \times X$ is open.
(ii) Show that φ is a regular map.