MATH 7141 - Algebraic Geometry I
Homework 6

Abstract varieties; finite maps ; fiber products

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume $\mathbb{K}$ is a field, unless otherwise indicated.

Problem 1. Show that for any $n \geq 1$, the projective space $\mathbb{P}^n_{\mathbb{K}}$ is an abstract variety.

Problem 2. Fix two abstract prevarieties X and Y over $\mathbb{K}$. Show that the Zariski topology on the product variety $X \times Y$ is finer than the product topology.

Given a closed subset Y of a prevariety X, we define the presheaf $\mathcal{O}_Y$ on Y as:

$$\mathcal{O}_Y(U) = \{\varphi\colon U \to \mathbb{K} : \forall\, a \in U,\ \exists\, V \subseteq X \text{ open and } \psi \in \mathcal{O}_X(V) \text{ with } \varphi \equiv \psi \text{ on } U \cap V\}$$

for every open U of Y. Then next problem shows that $(Y, \mathcal{O}_Y)$ is an abstract prevariety.

Problem 3. Fix a closed subset Y of a prevariety X.

(i) Show that the presheaf $\mathcal{O}_Y$ is a sheaf.

(ii) Show that if X is affine, then $(Y, \mathcal{O}_Y)$ is an affine variety (i.e., $\mathcal{O}_Y$ is the sheaf of regular functions on Y).

(iii) Using the previous item, show that for every affine open $U \subseteq X$ the ringed space $(U \cap Y, \mathcal{O}_{Y|_{U\cap Y}})$ is isomorphic to the affine variety $(U \cap Y, \mathcal{O}_{U\cap Y})$.

Problem 4. Show that the affine line with a double point is not a variety (i.e. show that the diagonal is not closed in the product).

Problem 5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be prevarieties, and consider an open cover

$$X = \bigcup_{i\in I} U_i,$$

with $\alpha_i\colon U_i \to X$ be the inclusion maps. Show that a map $\psi\colon X \to Y$ is a regular morphisms of prevarieties if, and only if, $\psi \circ \alpha_i\colon U_i \to Y$ is.

Problem 6. Let $X_1, \ldots, X_r$ be prevarieties with open subsets $U_{i,j} \subseteq X_i$ and isomorphisms $\psi_{i,j}\colon U_{i,j} \to U_{j,i}$ for each $i, j \in \{1, \ldots, r\}$ satisfying the required conditions for gluing $X_1, \ldots, X_r$ as in Lecture 24.

Assume Y is another prevariety with an open cover $Y = V_1 \cup \ldots \cup V_r$ and isomorphisms $g_i \colon V_i \to X_i$ satisfying

$$V_i \cap V_j = g_j^{-1}(U_{i,j}) \quad \text{and} \quad \psi_{i,j} \circ g_i = g_j \text{ on } V_i \cap V_j \quad \text{for all } i, j \in \{1, \ldots, r\}.$$

Show that there is a unique isomorphism $h \colon X \to Y$ such that $h(U_i) = V_i$ and $g_i \circ h = \text{inc} \colon U_i \hookrightarrow X_i$ for all $i = 1, \ldots, r$.

Problem 7. Consider two prevarieties X, Y. If X and Y are irreducible, show the same is true for $X \times Y$. (*Hint:* Show that the statement is true when X and Y are quasi-affine varieties, i.e. open subsets of affine varieties)

Problem 8. Let $\psi \colon X \to Y$ be a regular map between the affine varieties X and Y over an algebraically closed field $\mathbb{K}$. Write $\varphi = \psi_Y^\sharp \colon \mathbb{K}[Y] \to \mathbb{K}[X]$.

(i) Given any closed subset $W \subset X$, show that $I_Y(\overline{\psi(W)}) = \varphi^{-1}(I_X(W))$.
(ii) Prove that ψ is dominant if, and only if, φ is injective.

Problem 9. Let $\psi \colon X \to Y$ be a dominant regular map between two affine varieties over an algebraically closed field $\mathbb{K}$. Show that ψ is finite if, and only if, the ring homomorphism

$$\psi^\sharp_{D(f)} \colon \mathcal{O}_Y(D(f)) \to \mathcal{O}_X(\psi^{-1}(D(f)))$$

is finite for all $f \in \mathcal{O}_Y(Y) \smallsetminus \{0\}$.

Definition: A subset V of a topological space X is *locally closed* if for every $x \in V$, there is an open neighborhood U_x of x in X such that $U_x \cap V$ is closed in U_x. Equivalently: V is locally closed if, and only if, we can write V as $V = U \cap F$ where $U \subseteq X$ is open and $F \subseteq X$ is closed.

Problem 10. Show that given an abstract prevariety X, the diagonal $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ is a locally closed set. (*Hint*: Think about what happens for affine varieties.)

Problem 11. Show that a locally closed subset of a prevariety is a prevariety, by explicitly constructing its sheaf of regular functions (*Hint:* Use the definition for closed prevarieties and glue them together.)

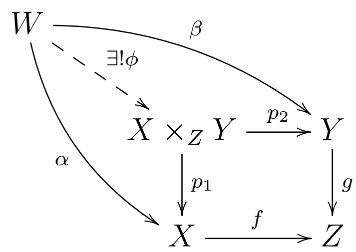
Problem 12. Given three prevarieties X, Y, Z together with two morphism $f \colon X \to Z$ and $g \colon Y \to Z$. Consider the set

$$H := \{(x, y) \in X \times Y \colon f(x) = g(y)\} \subseteq X \times Y$$

(i) Show that H is a locally closed subset of $X \times Y$. What is the sheaf of regular function on H? (*Hint:* Use Problem 11.)
(ii) Show that the natural projections $p_1 \colon H \to X$ and $p_2 \colon H \to Y$ are morphisms of prevarieties.

Definition: Given three prevarieties X, Y, Z with morphisms $f \colon X \to Z$ and $g \colon Y \to Z$, we define the *fiber product* $(X \times_Z Y, p_1, p_2)$ as the triple consisting of a prevariety and morphisms

$p_1\colon X\times_Z Y \to X$, $p_2\colon X\times_Z Y\to Y$ with $g\circ p_2 = f\circ p_1$ on W satisfying the following universal property. Given any prevariety W with morphisms $\alpha\colon W\to X$ and $\beta\colon W\to Y$ with $f\circ\alpha = g\circ\beta$ on W there exists a unique $\phi\colon W\to X\times_Z Y$ with $p_1\circ\phi=\alpha$ and $p_2\circ\phi=\beta$.



Problem 13. Show that fiber products exists in the category of prevarieties over algebraically closed fields (*Hint:* Show that the set H from Problem 12 satisfies the universal property of fiber products. You will need to use the universal property of products of prevarieties).

Problem 14. Fix three affine varieties X, Y, Z over and algebraically closed field $\mathbb{K}$ and $f\colon X\to Z$, $g\times Y\to Z$ be two regular maps. Show that

$$\mathbb{K}[X\times_Z Y] = (\mathbb{K}[X]\otimes_{\mathbb{K}[Z]}\mathbb{K}[Y])/\sqrt{(0)},$$

where $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ are viewed as $\mathbb{K}[Z]$-modules via the maps $f_Z^\sharp$ and $g_Z^\sharp$, respectively. Here, $\sqrt{(0)}$ is the nilradical of the zero ideal in the ring $\mathbb{K}[X]\otimes_{\mathbb{K}[Z]}\mathbb{K}[Y]$.

Problem 15. Let Z be the affine line with a double zero, and consider $X = Y = \mathbb{A}^1 \hookrightarrow Z$ the natural inclusions. Determine $X\times_Z Y$.

Problem 16. Show the following properties of fiber products of prevarieties over algebraically closed fields $\mathbb{K}$:

(i) Given X, Y, Z prevarieties and regular morphisms $f\colon X\to Z$ and $g\colon Y\to Z$, show that $X\times_Z Y\simeq Y\times_Z X$

(ii) Given X', X, Y, Z prevarieties with regular morphisms $f\colon X\to Z$, $g\colon Y\to Z$ and $f'\colon X'\to X$, show that $X'\times_X (X\times_Z Y)\simeq X'\times_Z Y$.

Problem 17. Given a morphism of prevarieties $f\colon X\to Z$ and a locally closed subset Y of Z, show that $X\times_Z Y\simeq f^{-1}(Y)$. What is the sheaf on $f^{-1}(Y)$?

Problem 18. Show that if X, Y, Z are varieties and $f\colon X\to Z$, $g\colon Y\to Z$ are morphisms of varieties, then $X\times_Z Y$ is a variety.

Problem 19. Consider two algebraic varieties X, Z defined over an algebraically closed field $\mathbb{K}$, and let $f\colon X\to Z$ be a finite map between them. Show that for every variety Y over $\mathbb{K}$ and every morphism $g\colon Y\to Z$ the induced morphism $p_2\colon X\times_Z Y\to Y$ is a finite morphism, by following the next two steps:

(i) Reduce the statement to the case when X, Y, Z are affine.

(ii) Assuming that X, Y, Z are affine, show that $(p_2^\sharp)_Y$ factors as

$$\mathcal{O}(Y) = \mathcal{O}(Z) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) \xrightarrow{f_Z^\sharp \otimes id} \mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) \xrightarrow{\pi} \mathcal{O}(X \times_Z Y) \ ,$$

where $\pi \colon \mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) \to \mathcal{O}(X \times_Z Y)$ is the quotient map obtained by taking the quotient by the nilradical, as in Problem 14.

(iii) Conclude from the previous item that the statement is true when X, Y, Z are affine.