MATH 7141 - Algebraic Geometry I
Homework 7

Dimension theory, tangent spaces, smoothness

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

In all problems below, we assume $\mathbb{K}$ is an arbitrary field, unless otherwise indicated.

In Lecture 30 we gave a definition of dimension of irreducible projective varieties X in $\mathbb{P}^n$ by means of linear projections from maximal linear subspaces of $\mathbb{P}^n$ disjoint from X. The next two exercises involve such construction.

Problem 1. Let X be the twisted cubic in $\mathbb{P}^2_{\mathbb{K}}$ defined in Problem 2 of Homework 2. Assume $\mathbb{K}$ is algebraically closed.

(i) Find a linear subspace $\Lambda = \mathbb{P}(W)$ in $\mathbb{P}^2 = \mathbb{P}(\mathbb{K}^3)$ of maximal dimension disjoint from X and compute the linear projection $\pi\colon X \to Y = \mathbb{P}(\mathbb{K}^3/W)$, as defined in Lecture 30.

(ii) Show that π is a finite morphism by explicitly computing the morphism

$$\varphi = \pi^\sharp_Y\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X).$$

Problem 2. Let X be the Veronese surface in $\mathbb{P}^5$ over an algebraically closed field $\mathbb{K}$.

(i) Find a linear subspace $\Lambda = \mathbb{P}(W)$ in $\mathbb{P}^2 = \mathbb{P}(\mathbb{K}^3)$ of maximal dimension disjoint from X and compute the linear projection $\pi\colon X \to Y = \mathbb{P}(\mathbb{K}^3/W)$, as defined in Lecture 30.

(ii) Show that π is a finite morphism by explicitly computing the morphism

$$\varphi = \pi^\sharp_Y\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X).$$

Problem 3. Fix $X \subset \mathbb{A}^n$ an irreducible affine variety over an algebraically closed field $\mathbb{K}$. Consider $\overline{X} \subset \mathbb{P}^n$, a linear subspace $\Lambda = \mathbb{P}(W)$ in $\mathbb{P}^2 = \mathbb{P}(\mathbb{K}^{n+1})$ of maximal dimension disjoint from $\overline{X}$ and the linear projection $\pi\colon \overline{X} \to \mathbb{P}(\mathbb{K}^{n+1}/W) = \mathbb{P}^r$, as defined in Lecture 30. Show that the restriction of π to X yields a finite surjective morphism $X \to \mathbb{A}^r$.

Problem 4. Given three irreducible affine varieties X_1, X_2, X_3 over an algebraically closed field $\mathbb{K}$ with $X_1 \subseteq X_2 \subseteq X_3$, show that

$$\operatorname{codim}_{X_3} X_1 = \operatorname{codim}_{X_2} X_1 + \operatorname{codim}_{X_3} X_2.$$

Problem 5. Let X be a topological space and $U \subseteq X$ an open subset. Let $Y \subseteq X$ be an irreducible closed set with $U \cap Y \neq \emptyset$. Show that $U \cap Y$ is irreducible and closed in U and, furthermore, we have

$$\operatorname{codim}_X(Y) = \operatorname{codim}_U(U \cap Y).$$

Problem 6. Show that if X and Y are quasi-affine varieties, then

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

Recall from Problem 10 Homework 6 the definition of locally closed sets.

Problem 7. Let X be a quasi-affine variety and Z a locally closed subset of X. Show that

$$\dim(Z) = \dim(\overline{Z}) > \dim(\overline{Z} \smallsetminus Z).$$

Problem 8. Show that if X and Y are irreducible affine varieties of $\mathbb{A}^n$, then every irreducible component Z of $X \cap Y$ satisfies:

$$\dim(Z) \geq \dim(X) + \dim(Y) - n.$$

(*Hint:* Describe $X \cap Y$ as the intersection of $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^n$ with the diagonal $\Delta = \{(x, x) : x \in \mathbb{A}^n\}$.)

Given an abstract variety X over an algebraically closed field $\mathbb{K}$, and a point p in X, we define the *local dimension* of X at p as the quantity:

$$\dim_p(X) := \dim \mathcal{O}_{X,p}.$$

Problem 9. Let X be an abstract variety over an algebraically closed field $\mathbb{K}$, and fix a point p in X. If $X = X_1 \cup \ldots \cup X_s$ is the irreducible decomposition of X, show that

$$\dim_p X = \max_{1 \leq i \leq s} \{\dim_p X_i : p \in X_i\}.$$

(*Hint:* Reduce the computation to the case when X is affine, by working with an affine open cover of X. Prove the statement for the affine case, by identifying a chain of prime ideals in $\mathcal{O}_{X,p}$ with a chain of prime ideals in $\mathbb{K}[X]$.)

For the following exercise, use the results from Problems 11 and 12 in Homework 2:

Problem 10. Find $\dim V(I)$ for the following monomial ideals:

(i) $I = \langle xy^3z, x^2 \rangle \subseteq \mathbb{C}[x, y, z]$;
(ii) $I = \langle wx^2z, w^3y, wxyz, x^5y^4 \rangle \subseteq \mathbb{C}[x, y, z, w]$.

Problem 11. Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal, where $\mathbb{K}$ is an arbitrary closed field. Show that $\dim(V(I)) = 0$ if and only if for each $i \in \{1, \ldots, n\}$ we can find $m_i \geq 1$ with $x_i^{m_i} \in I$.

Problem 12. Fix $X \subseteq \mathbb{P}^n$ a projective variety over an algebraically closed field $\mathbb{K}$, and let $p \in X$ be any point. Pick $i \in \{0, \dots, n\}$ such that p lies in the affine patch U_i. Show that the projective tangent space $T_pX \subseteq \mathbb{P}^n$ (which contains the point p) satisfies:

$$(T_pX) \cap U_i = T_{\overline{p}}(X \cap U_i) + \overline{p}.$$

Here, $\overline{p}$ corresponds to the image of p under the isomorphisms $U_i \simeq \mathbb{A}^n$, and $T_{\overline{p}}(X \cap U_i)$ denotes the tangent space of the affine variety $X \cap U_i \subseteq \mathbb{A}^n$ at the point $\overline{p}$. (*Hint:* you may assume $p \in U_0$ up to reordering the coordinates of $\mathbb{P}^n$.)

Problem 13. Given two varieties X and Y and points $x \in X, y \in Y$, the projection maps $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ induce a linear map:

$$T_{(x,y)}(X \times Y) \to T_xX \times T_yY.$$

Write down this map explicitly and show that it is an isomorphism.

Problem 14. Find all singular points of the following affine varieties in $\mathbb{A}_{\mathbb{K}}$. where $\mathbb{K}$ is an algebraically closed field:

(i) $X = V(y^2 - x^3 + 3) \subseteq \mathbb{K}^2$
(ii) $X = V(x^2y + x^3 + y^3) \subseteq \mathbb{K}^3$

Problem 15. Find all singular points of the following affine varieties in $\mathbb{A}_{\mathbb{K}}$. where $\mathbb{K}$ is an algebraically closed field:

(i) $X = V(x^2y^2 + x^2 + y^2 + 2xy(x + y + 1)) \subseteq \mathbb{K}^2$,
(ii) $X = V(x^3 - zxy + y^3) \subseteq \mathbb{K}^3$