Lecture I: Overview & Course description

§1.1 Overview:

Algebraic geometry is the study of solutions to polynomial equations over a field $\mathbb{K}$
(Study Geometry through Algebra of polynomials)

⚠ Questions of interest & techniques vary drastically depending on $\mathbb{K}$.

① $\boxed{\mathbb{K} = \mathbb{Q}\text{, alg. extn of } \mathbb{Q}\text{; finite fields, p-adics, function fields}}$

⇝ Arithmetic Questions (eg. solve diophantine equations)

- Example (Fermat) $x^n + y^n = z^n$ has no-nontrivial solutions over $\mathbb{Q}$. if $n \geq 3$.
- Counting rational points on X (over $\mathbb{F}_q$, alg extn of $\mathbb{Q}$, p-adics, $\mathbb{Q}(t)$, etc) &
- Finding one rat'l pt & develop techniques to get others from that one.

Central problem: Weil Conjectures (1949) on the generating function (local zeta function) derived from counting pts on algebraic varieties over finite fields.

$$\zeta(X, s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} q^{-ms} \right)$$

X projective alg var / $\mathbb{F}_q$

$N_m = \# X(\mathbb{F}_{q^m})$

Conjecture
(1) rational function of $T = q^{-s}$ (Thm by Dwork (1960))
(2) Explicit functional equation (Thm by Grothendieck (1965))

$$\zeta(X, n-s) = \pm q^{nE/2 - Es}\, \zeta(X,s) \qquad E = \text{Euler char of } X$$

(3) Restricted location of its zeros (Thm by Deligne (1974))

Example: $X = \mathbb{P}^1_{\mathbb{F}_q}$, $\zeta(X,s) = \dfrac{1}{(1-q^{-s})(1-q^{1-s})}$

Proofs via a suitable cohomology theory.

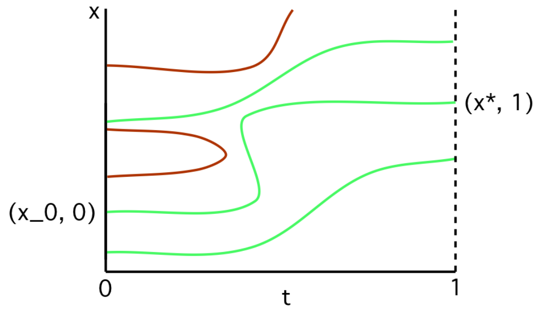
② $\boxed{\mathbb{K} = \mathbb{R}\text{, or a real closed field (equiv. by Tarski principle)}}$

- Most used for Applications. (eg CAD, Robotics)

• Main latest development: Homotopy Continuation, a numerical method to compute solutions of polynomial systems over $\mathbb{R}$ with finite solution sets

Idea: (1) deform systems to simpler ones to solve

(2) Track solutions as we deform back, hoping for numerical stability



Extracted from "Homotopy continuation" by Max Buot & Donald Richards. Talk delivered at the Summer School in Statistics for Astronomers & Physicists (16 June 2005)

https://astrostatistics.psu.edu/su05/max_homotopy061605.pdf

• Answers depend on the Topology of the real set.

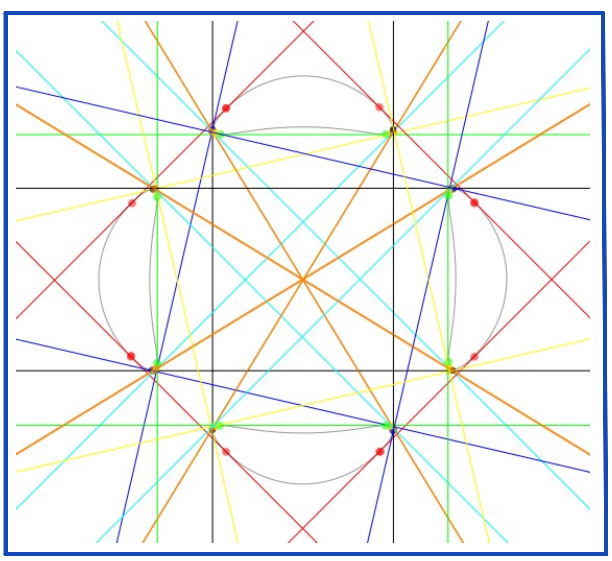
Pluecker: (1834) A smooth quartic curve in $\mathbb{P}^2$ (solutions to a homogeneous eqns in 3 variables of degree 4) over $\mathbb{C}$ has exactly 28 bitangent lines

• Bitangent = line tangent to the $\mathbb{C}$-curve at 2 points

• Real bitangent = equation of the bitangent is defined over $\mathbb{R}$ (tangency pts could be complex!)

Zeuthen (1873) Over $\mathbb{R}$, the number depends on the Topology of the real curve.

Topology	4 ovals	3 ovals	2 non-nested ovals	2 nested ovals	1 oval	$\emptyset$
# Real bitangents	28	16	8	4	4	4



Example by Trott of a real quartic with 4 ovals and its 28 real bitangents.

Source: https://en.wikipedia.org/wiki/Bitangents_of_a_quartic#/media/File:TrottCurveBiTangents28.svg

③ $\boxed{\mathbb{K} = \mathbb{C} \quad \text{(or algebraically closed fields)}}$

Many more Tools: algebra, differential geometry & $\mathbb{C}$-analysis

Questions
- geometric invariants (dimension, homology / cohomology, Euler charact, ...)
- decide if singular / smooth & "resolve" singularities (effectively)
- decide if compact; find nice compactifications
- Can we decompose varieties? (Primary decompositions)
- Can we compute invariants explicitly? (Computational Tools, eg Gröbner basis)
- Can we classify varieties? Can we study families, eg moduli spaces?
- Enumerative questions (eg "27 Lines on smooth cubic surfaces on $\mathbb{P}^3_{\mathbb{C}}$")
- Can we construct special varieties?

§1.2 Course Outline:

In this course we will focus on learning the theory by means of classical examples. Whenever in need, we'll focus on $\mathbb{K} = \overline{\mathbb{K}}$ (& even $\mathbb{K} = \mathbb{C}$ if necessary)

Course outline: (1) Affine Varieties (Weeks 1-3) Issue: NOT compact!

(2) Projective Varieties (Weeks 4-6) (Compact!)

(3) Interlude on general Theory: Sheaves, Dimension, Function fields, Smoothness, Resolution of Singularities, Divisors, Line Bundles, Cohomology, Gröbner bases (Weeks 7-11)

(4) Curve Theory (Weeks 12-15)

(5) del Pezzo surfaces (Week 16)

- Missing: Topic from the year-long sequence: Varieties coming from Combinatorics (eg Toric / Tropical Varieties)

(Future Topics course?)