Lecture II: Affine varieties, first examples & Zariski Topology

§1. Introduction to affine varieties

Fix a field $\mathbb{K}$ ($=\mathbb{C}$ to fix ideas)

Definition: The n-dimensional affine variety $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ is the set $\mathbb{K}^n$.

(Think of $\mathbb{A}^n$ as a set of points without the additional $\mathbb{K}$-vector space structure)

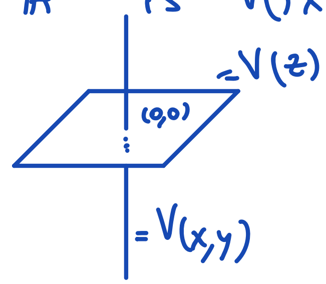
Definition: An algebraic subset of $\mathbb{A}^n$ or an affine subvariety of $\mathbb{A}^n$ is the zero locus of a collection S of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$.

$$\boxed{V(S) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f(a_1, \ldots, a_n) = 0 \quad \forall f \in S\}}$$

↳ for vanishing locus.

Examples:

0. $\emptyset = V(\{1\})$
1. $\mathbb{A}^n_{\mathbb{K}} = V(\{0\})$
2. Line $\mathbb{A}^1$  in $\mathbb{A}^2$ is $V(\{x\}) = V(\{x^2\})$

3. $V(xz, yz)$ in $\mathbb{A}^3$ = 

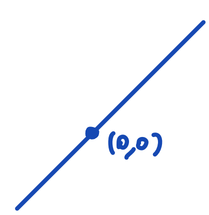
2 components (1 of dim 1 = line, 1 of dim 2 = plane) (dim as vector spaces)

Q: What is the dimension of this affine variety?

A: $\max\{\dim(\text{components})\} = \max\{1,2\} = 2$ (future lecture)

Remark: Ex 3 is reducible $V = V_1 \cup V_2$ with $V_i \subsetneq V$ for $i=1,2$.

⚠ Some books assume varieties to be irreducible, so be careful when reading other sources!

Examples: 4. $V(x(x-y), y(x-y))$ in $\mathbb{A}^2$ =  = line ∪ pt where

the point is embedded in the line. As a set, we'll record the line $= V(x-y)$ AND the embedded point $(0,0) = V(x,y)$. This variety is reducible!

The next lemmas show that $V(\ )$ is an inclusion-reversing operation & that varieties are closed under finite unions & arbitrary intersections.

Lemma 1: If $S_1 \subseteq S_2 \subseteq \mathbb{K}[x_1,\dots,x_n]$ we have $V(S_1) \supset V(S_2)$

Lemma 2: If $S_1, S_2 \subseteq \mathbb{K}[x_1,\dots,x_n]$ we have

$$V(S_1) \cup V(S_2) = V(S_1 S_2)$$

where $S_1 \cdot S_2 = \{fg : f \in S_1 \ \& \ g \in S_2\}$

Proof: ($\subseteq$) $\underline{a} \in V(S_1) \Rightarrow f(\underline{a}) = 0 \quad \forall f \in S_1$

$\Rightarrow f \cdot g = 0 \quad \forall f \in S_1 \ \& \ \forall g \in S_2$, hence $\underline{a} \in V(S_1 S_2)$

. Same ideas give $V(S_2) \subseteq V(S_1 S_2)$.

($\supseteq$) We prove the contrapositive. Say $\underline{a} \notin (V(S_1) \cup V(S_2))$.
So $\exists f \in S_1, g \in S_2$ with $f(\underline{a}) \neq 0$ & $g(\underline{a}) \neq 0$. But then $(fg)(\underline{a}) \neq 0$, meaning $\underline{a} \notin V(S_1 S_2)$ □

Lemma 3: If Λ is any index set & $S_i \subseteq \mathbb{K}[x_1,\dots x_n]$ for any $i \in \Lambda$,
then

$$\bigcap_{i \in \Lambda} V(S_i) = V\Big(\bigcup_{i \in I} S_i\Big)$$

Proof: $\underline{a} \in \bigcap_{i \in \Lambda} V(S_i) \iff \forall i \ \& \ \forall f \in S_i$ we have $f(\underline{a}) = 0$.

$\iff \forall f \in \bigcup_{i \in \Lambda} S_i$ we have $f(\underline{a}) = 0 \iff \underline{a} \in V\Big(\bigcup_{i \in \Lambda} S_i\Big)$.

Proposition 1: $V(S) = V(\langle S \rangle)$ where $\langle S \rangle =$ ideal in $\mathbb{K}[\underline{x}]$ generated by S.

Proof: We prove the double inclusion.

($\supseteq$) $S \subseteq \langle S \rangle$ so by Lemma 1 we have $V(S) \supseteq V(\langle S \rangle)$.

($\subseteq$) Say $f \in \langle S \rangle$ so $\exists k \geq 1, \exists f_1, \dots, f_k \in S$ & $g_1, \dots, g_k \in \mathbb{K}[\underline{x}]$ with $f = \sum_{i=1}^{k} g_i f_i$

If $\underline{a} \in V(S)$, then $f_i(\underline{a}) = 0 \quad \forall i = 1, \dots, k$. In particular,

$$f(\underline{a}) = \Big(\sum_{i=1}^{k} g_i f_i\Big)(\underline{a}) = \sum_{i=1}^{k} g_i(\underline{a}) f_i(\underline{a}) = \sum_{i=1}^{k} g_i(\underline{a}) \cdot 0 = 0$$

Conclusion: $f(\underline{a})=0 \quad \forall f \in \langle S \rangle$ ie $\underline{a} \in V(\langle S \rangle)$ $\square$

Corollary 1: Any affine subvariety of $\mathbb{A}^n$ is defined by a finite list of polynomials

Proof This is a direct consequence of Hilbert's Basis Thm ($\mathbb{K}[x_1, \ldots, x_n]$ is a Noetherian ring, so every ideal is finitely generated).

Indeed $V(S) = V(\langle S \rangle)$ by Proposition 1. Pick generators $f_1, \ldots, f_k$ for the ideal $\langle S \rangle$. Then, $V(S) = V(\langle f_1, \ldots, f_k \rangle) = V(\{f_1, \ldots, f_k\})$ $\square$

Corollary 2: $\displaystyle\bigcap_{j \in \Lambda} V(S_j) = V(\underbrace{\langle S_{j_1} \rangle + \cdots + \langle S_{j_k} \rangle}_{\text{sum of ideals}})$ for some choice of indices $j_1, \ldots, j_k \in \Lambda$

§2 Interlude: Zariski Topology:

Lemmas 2 & 3, together with examples 0 & 1 confirm that we can build a topology on $\mathbb{A}^n$, called the Zariski Topology (closed sets = affine subvar of $\mathbb{A}^n$)

Definition: We say $U \subseteq \mathbb{A}^n$ is an open set of $\mathbb{A}^n$ in the Zariski Topology if $V = \mathbb{A}^n \setminus U$ is an affine variety

Proposition 3: $\{\mathbb{A}^n \setminus V(S)\}_{S \subseteq \mathbb{K}[\underline{x}]}$ defines a topology on $\mathbb{A}^n$

Proof: Direct from the axioms defining a topology.

Example: Zariski closed sets in $\mathbb{A}^1 = V(S) = V(\langle S \rangle) = V(f)$ because $\mathbb{K}[x]$ is a PID

So we have 3 options: $\emptyset$ ($f \in \mathbb{K}$), $\mathbb{A}^1$ ($f=0$) or a finite set ($\deg f \neq 0, -\infty$).

Thus, opens are $\emptyset$ or complements of finite sets, ie the Zariski Top on $\mathbb{A}^1_{\mathbb{K}}$ is the cofinal topology.

⚠ If $\mathbb{K}=\mathbb{C}$ we can endow $\mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$ with the Euclidean Topology

These 2 topologies are different! A Zariski closed set is closed in the Euclidean Topology (polynomials are continuous functions & $V(S) = \displaystyle\bigcap_{j=1}^{k} V(f_j)$ for some finite collection $\{f_1, \ldots, f_k\}$) but not conversely!

For example : $\overline{B_{(0,1)}}$ cannot be closed in the Zariski Topology (Any polynomial vanishing on this set will be $=0$! This is easy to see for $n=1$ by the Fundamental Thm. of Algebra).

- Conclusion: Intuition built up from $\mathbb{K}=\mathbb{C}$ can sometimes be misleading!

- Using the inclusion $V \subseteq \mathbb{A}^n$ we can endow any affine subspace with its own Zariski Topology. More precisely, we use the subspace topology.

Definition: A set $U \subseteq V$ is open in the Zariski Topology if $U = V \cap \tilde{U}$ for some $\tilde{U} \subseteq \mathbb{A}^n$ open in the Zariski Top.

Obs: This choice will allow us to consider V as an ambient variety (as opposed to always consider $\mathbb{A}^n$ as the ambient space)

Lemma 4: Every Zariski open dense subset of $\mathbb{A}^n_{\mathbb{C}}$ is dense in the Euclidean Topology

Proof : Exercise

Remark: The statement fails if we drop the open condition: For example, an infinite discrete collection of points in $\mathbb{A}^1_{\mathbb{C}}$ (eg $\mathbb{Z}+i\mathbb{Z}$) is Zariski dense but not open. It is also not dense in the Euclidean Topology.

- We have a natural basis for the Zariski topology on $\mathbb{A}^n$ (smallest opens are the complements of largest proper varieties $= V(f)$ for some $f \in \mathbb{K}[x_1, \ldots, x_n]$)

Proposition 4: The collection $\mathcal{B} = \{D(f) = \mathbb{A}^n \setminus V(f) : f \in \mathbb{K}[x_1, \ldots x_n]\}$ defines a basis for the Zariski Topology. We call $D(f)$ an affine basic open.

Proof : Consequence of Corollary 1.

- The following result also confirms that the Zariski Top & the Euclidean Top on $\mathbb{A}^n_{\mathbb{C}}$ are difference:

Lemma 5: If $\mathbb{K}$ is infinite, the Zariski Top on $\mathbb{A}^n_{\mathbb{K}}$ is NOT Hausdorff.

Furthermore, any 2 open sets must intersect.

Proof: Pick $U_1 = \mathbb{A}^n \setminus V_1$ & $U_2 = \mathbb{A}^n \setminus V_2$ two Zariski opens. Then

$$U_1 \cap U_2 = (\mathbb{A}^n \setminus V_1) \cap (\mathbb{A}^n \setminus V_2) = \mathbb{A}^n \setminus (V_1 \cup V_2)$$

Since $V_1 = V(S_1) \subseteq V(f_1) \quad \forall f_1 \in \langle S_1 \rangle$
$V_2 = V(S_2) \subseteq V(f_2) \quad \forall f_2 \in \langle S_2 \rangle$ our statement will follow if

we show that $\mathbb{A}^n \neq V(f) \cup V(g)$ for any $f, g \in \mathbb{K}[x_1, \ldots, x_n] \setminus \{0\}$

We can prove this by induction on n:

- Base case: $n=1$ $\quad \mathbb{A}^1$ is infinite whereas $V(f)$ & $V(g)$ are finite if $f, g \notin \mathbb{K}$.
- Inductive step: Pick any $a \in \mathbb{K}$ & consider $\mathbb{A}^{n-1} = V(x_n - a)$.

We argue by contradiction. If $\mathbb{A}^n = V(f) \cup V(g)$, then

$$\begin{aligned} \mathbb{A}^{n-1} &= \big(V(f) \cap V(x_n - a)\big) \cup \big(V(g) \cap V(x_n - a)\big) \\ &= V(f(x', a)) \cup V(g(x', a)) \qquad \text{for } x' = (x_1, \ldots, x_n) \end{aligned}$$

By our IH we get either $f(x', a) = 0$ or $g(x', a) = 0$. but this can only happen if $x_n - a \mid f$ or $x_n - a \mid g$. (because $\mathbb{K}$ is infinite)

Since a was arbitrary, this cannot happen. Conclusion: $\mathbb{A}^n \neq V(f) \cup V(g)$ $\square$

§3 Zariski vs Product Topology:

Note $\mathbb{A}^n = \mathbb{A}^{n-k} \times \mathbb{A}^k$ as sets for any $k = 1, \ldots, n-1$.

Thus, we can endow $\mathbb{A}^n$ with a new Topology, the product Top coming from the right side. This is the coarsest one making the projections

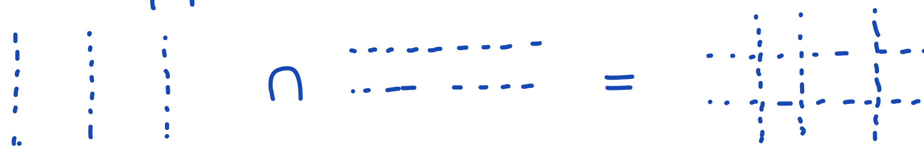
$$p_1 : \mathbb{A}^{n-k} \times \mathbb{A}^k \longrightarrow \mathbb{A}^{n-k} \qquad \& \qquad p_2 : \mathbb{A}^{n-k} \times \mathbb{A}^k \longrightarrow \mathbb{A}^k$$
$$(\underline{x}, \underline{y}) \longmapsto \underline{x} \qquad\qquad\qquad (\underline{x}, \underline{y}) \longmapsto \underline{y}$$

continuous.

Q: Is this Topology the same as the Zariski one? A: NO!

- We can see this already for $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$

- A <u>basis</u> for the product top on $\mathbb{A}^1 \times \mathbb{A}^1$ is given by finite intersections of sets of the form $p_i^{-1}(U_i)$ where $U_i \subseteq \mathbb{A}^1$ is Zariski open, ie $p_1(U_1) \cap p_2(U_2)$ where U_i are $\emptyset$ or complements of finite sets in $\mathbb{A}^1$.

  Pictorially: Opens  , so opens in the product top are complements of finitely many horiz & vertical lines ("cylinders")

- $V(x-y) \subseteq \mathbb{A}^2$ is Zariski closed, but it is <u>not</u> closed in the product topology

  It's complement is not an intersection of cylinders.