Lecture III: Ideals of varieties, Basic duality & irreducible decompositions

Recall: $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ is an algebraic subvariety if $V = V(S) = \{\underline{a} \in \mathbb{K}^n : f(\underline{a}) = 0 \ \forall f \in S\}$

Lemma: (1) $V(S_1 \cap S_2) = V(S_1) \cup V(S_2)$ for $S_1, S_2 \subseteq \mathbb{K}[x_1, \ldots x_n]$

(2) $\bigcap_{i \in \Lambda} V(S_i) = V\left(\bigcup_{i \in I} S_i\right)$ $\forall S_i \subseteq \mathbb{K}[x_1, \ldots, x_n]$

(3) $\emptyset = V(\{1\})$; $\mathbb{A}^n = V(0)$.

(4) $V(S) = V(\langle S \rangle)$ so we can assume S is finite

(5) $S_1 \subseteq S_2 \implies V(S_1) \supseteq V(S_2)$ ("inclusion reversing")

Corollary: Varieties are closed sets for a topology (Zariski Top)

TODAY: Focus on ideals constructed from affine subvarieties of $\mathbb{A}^n$.

§1. Ideals from affine varieties:

Definition: Given a subset $W \subseteq \mathbb{A}^n$, we define:

$$\boxed{I(W) = \{f \in \mathbb{K}[x_1, \ldots, x_n] : f(\underline{a}) = 0 \quad \forall \underline{a} \in V\} \subseteq \mathbb{K}[x_1, \ldots, x_n]}$$

(This makes sense even if W is not an affine variety)

Proposition 1: $I(W)$ is an ideal of $\mathbb{K}[\underline{x}]$

Proof: Need to show 3 properties of ideals:

(1) $0 \overset{?}{\in} I(W)$: $0(\underline{a}) = 0 \quad \forall \underline{a} \in V$ ✓

(2) $f, g \in I(W) \overset{?}{\implies} f+g \in I(V)$

$f(\underline{a}) = 0$ & $g(\underline{a}) = 0$ $\forall \underline{a} \in W$ by def, so $(f+g)(\underline{a}) = f(\underline{a}) + g(\underline{a}) = 0+0=0$ $\forall \underline{a} \in W$ ✓

(3) $f \in I(W)$, $h \in \mathbb{K}[\underline{x}] \overset{?}{\implies} h \cdot f \in I(W)$

$f(\underline{a}) = 0$ $\forall \underline{a} \in W$, so $(hf)(\underline{a}) = h(\underline{a})\, f(\underline{a}) = h(\underline{a}) \cdot 0 = 0$ $\forall \underline{a} \in W$ ✓ □

We have the analogs of Lemmas 1 & 2 for §1.1:
[(1) & (5) above]

Lemma 1: If $W_1 \subseteq W_2$, then $I(W_1) \supseteq I(W_2)$.

Lemma 2: If W_1, W_2 are subsets of $\mathbb{A}^n$ we have:
$$I(W_1 \cup W_2) = I(W_1) \cap I(W_2)$$

Proof: $f \in I(W_1 \cup W_2) \iff \forall \underline{a} \in W_1 \cup W_2$ we have $f(\underline{a}) = 0$
$\iff \forall \underline{a} \in W_1$ we have $f(\underline{a}) = 0$ & $\forall \underline{a} \in W_2$ we have $f(\underline{a}) = 0$
$\iff f \in I(W_1)$ & $f \in I(W_2) \iff f \in I(W_1 \cap W_2)$ $\square$

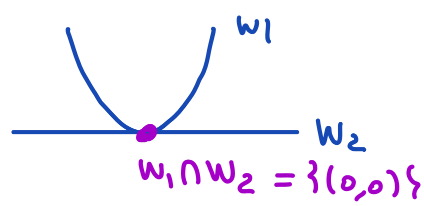
We have a weaker version of Lemma 3 from §1.1:

Lemma 3: If W_1, W_2 are subsets of $\mathbb{A}^n$ we have:
$$I(W_1 \cap W_2) \supseteq I(W_1) + I(W_2)$$

Proof: Since $I(W_1 \cap W_2)$ is an ideal, it suffices to check that $I(W_i) \subseteq I(W_1 \cap W_2)$ for $i = 1, 2$, but this follows from Lemma 1 since $W_i \supseteq W_1 \cap W_2$ $\square$

⚠ The inclusion $\supseteq$ can be strict even if W_1 & W_2 are affine varieties.

Example: $W_1 = V\langle y - x^2 \rangle$
$W_2 = V\langle y \rangle$



- $I(W_1 \cap W_2) = V(x, y)$ (any poly vanishing on $(0,0)$ has constant term $= 0$, so it lies in the ideal $\langle x, y \rangle$)
- $I(W_1) = \langle y - x^2 \rangle$ (any $f \in I(W_1)$ can be viewed in $K[x][y]$ & we can replace f by $f = g(x) + h(x,y)(y - x^2)$ by replacing any y with $(y - x^2) + x^2$. & reordering)
- $I(W_2) = \langle y \rangle$ (similar idea: Write $f \in I(W_2)$ as $f = f(x) + h(x,y)\, y$ & conclude $f(x) = 0$ because it vanishes on $\mathbb{A}^1 = V(y)$).

- $I(W_1) + I(W_2) = \langle y - x^2 \rangle + \langle y \rangle = \underset{x \notin}{\langle y, x^2 \rangle} \neq \langle y, x \rangle.$

§2. Duality between ideals & varieties:

The results from §1.1 & §2.1 yield the following Basic Duality for affine subvar. of $\mathbb{A}^n$

$$\boxed{\begin{array}{ccc} \text{GEOMETRY} & & \text{ALGEBRA} \\ \left\{\begin{array}{c}\text{Subvarieties} \\ \text{of } \mathbb{A}^n\end{array}\right\} & \begin{array}{c} \xrightarrow{\ \ I\ \ } \\ \xleftarrow[\ \ V\ \ ]{} \end{array} & \left\{\begin{array}{c}\text{ideals } \mathfrak{a} \\ \text{of } \mathbb{K}[x_1, \ldots, x_n]\end{array}\right\} \end{array}}$$

Next, we discuss how I & V interact with each other.

Proposition 2: If $W \subseteq \mathbb{A}^n$ is a subvariety, then $V(I(W)) = W$.

Proof: ($\supseteq$) is easy to check: If $\underline{a} \in W$, then $f(\underline{a}) = 0$ $\forall f \in I(W)$ by definition of $I(W)$, meaning $\underline{a} \in V(I(W))$

($\subseteq$) is also easy to check. Since W is algebraic, then $W = V(S)$ for a finite set $S = \{f_1, \ldots, f_k\}$ of polynomials in $\mathbb{K}[x_1, \ldots x_n]$ (Corollary 1 §1.1)
We have $S \subseteq I(W)$ by definition of S, so by Lemma 1 we have

$$W = V(S) \supseteq V(I(W)) \qquad \square$$

Corollary 1: For 2 varieties W_1 & W_2 we have $W_1 \subseteq W_2 \iff I(W_1) \supseteq I(W_2)$

Proof: Combine Prop 1 §2.1 & Lemma 3 §1.1

Proposition 2: For any ideal $\mathfrak{a}$ of $\mathbb{K}[x_1, \ldots, x_n]$ we have $I(V(\mathfrak{a})) \supseteq \mathfrak{a}$

Proof: Pick any $\underline{a} \in W = V(\mathfrak{a})$ & $f \in \mathfrak{a}$. Then, $f_{(\underline{a})} = 0$ & so $f \in I(V(\mathfrak{a}))$
In particular, $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$. $\square$

⚠ This is **not** a 1-to-1 correspondence even if $\mathbb{K}=\overline{\mathbb{K}}$. (ideals are restricted!)

Ex: $\mathbb{A}^1 = V(\{x\}) = V(\{x^2\})$

Lemma 4: $I(W)$ is a radical ideal for any $W \subseteq \mathbb{A}^n$.

(J is radical if for all f with $f^N \in J$ for some $N \Rightarrow f \in J$)

Proof: If $f^N \in I(W)$ then $(f^N)(\underline{a}) = (f(\underline{a}))^N = 0 \quad \forall \underline{a} \in W$. But $\mathbb{K}$ is a field, so this forces $f(\underline{a})=0 \quad \forall \underline{a} \in W$, ie $f \in I(W)$ $\square$

Corollary 2: For any ideal $\mathfrak{a} \subseteq \mathbb{K}[x_1,\ldots,x_n]$, we have $I(V(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}$.

⚠ Inclusion can be strict!

Example: $\mathfrak{a} = \langle 1+x^2 \rangle \subseteq \mathbb{R}[x]$ Then $V(\mathfrak{a}) = \emptyset$ ($\mathbb{K}=\mathbb{R}$)

& $I(V(\mathfrak{a})) = I(\emptyset) = \mathbb{R}[x] \neq \sqrt{\langle 1+x^2 \rangle} = \langle 1+x^2 \rangle$

↳ $1+x^2$ is square free!

• In $\mathbb{K}=\overline{\mathbb{K}}$ we can do better!

Hilbert's Nullstellensatz (Strong version): If $\mathbb{K}=\overline{\mathbb{K}}$ & $\mathfrak{a} \subseteq \mathbb{K}[x_1,\ldots,x_n]$ is an ideal, then

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

In particular, when $\mathfrak{a}=(1)$, we get:

Weak Hilb. Nullstellensatz: If $\mathbb{K}=\overline{\mathbb{K}}$ & $\mathfrak{a}$ is an ideal of $\mathbb{K}[x_1,\ldots,x_n]$ we have:

$$V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = (1) = \mathbb{K}[x_1,\ldots,x_n]$$

Remark: We'll see that Weak Hilbert Nullstellensatz $\Longrightarrow$ Strong Hilbert Nullstellensatz

Corollary 3: If $\mathbb{K}=\overline{\mathbb{K}}$, affine subvarieties are in 1-to-1 corresp. to radical ideals under the maps $I(\cdot)$ & $V(\cdot)$.

Q: What to do about the basic duality when $\mathbb{K}$ is an arbitrary field?

A: The Theory of Schemes (Math 7142) will arise by enlarging the Geometric side to match all ideals (tautologically!)

- Ideals will correspond to "affine schemes"
- Schemes will be obtained by glueing affine schemes

$$\boxed{\begin{array}{c}\text{Local picture} \longleftrightarrow \text{Commutative Algebra ;} \\ \text{Global picture} \longleftrightarrow \text{Homological Alg.}\end{array}}$$

Next week, we'll discuss a proof of Hilbert's Nullstellensatz (2 statements)
We'll need some Commutative Algebra.