Lecture IV: Irreducible Decomposition of Varieties
Primary Ideals of Noetherian commutative rings

Recall Basic duality in Algebraic Geometry

GEOMETRY		ALGEBRA
$\left\{ \begin{array}{c} \text{Affine Subvarieties} \\ W \text{ of } \mathbb{A}^n_{\mathbb{K}} \end{array} \right\}$	$\xrightarrow{I}$ $\xleftarrow[V]{}$	$\left\{ \begin{array}{c} \text{radical ideals } \mathfrak{a} \\ \text{of } \mathbb{K}[x_1, \ldots, x_n] \end{array} \right\}$

Lemma: (1) $W \subseteq \mathbb{A}^n$ subvariety, then $V(I(W)) = W$

(2) $W_1, W_2 \subseteq \mathbb{A}^n$ subvarieties, then $W_1 \subseteq W_2 \iff I(W_1) \supseteq I(W_2)$

(3) $I(W_1 \cup \cdots \cup W_r) = I(W_1) \cap \cdots \cap I(W_r)$

§1. Irreducible decomposition of affine varieties:

Inspired by Example 3 §2.1, we define:

Definition: A variety $V \subseteq \mathbb{A}^n$ is irreducible if for every expression of V as a union $V = V_1 \cup V_2$ we have either $V_1 = V$ or $V_2 = V$.

A variety is reducible if it is not irreducible.

Example: ① $\{0\}$ is irreducible

② $V(xz, yz) = V(x,y) \cup V(z)$ in $\mathbb{A}^3$ is reducible.

Theorem 1: Every affine variety is a finite union of irreducible varieties.

Proof: We use a "bisection" argument. If a variety V is irreducible, then is nothing to do. If V is reducible, we write $V = V_1 \cup V_1'$ with $V_1 \subsetneq V$ & $V_1' \subsetneq V$

- If both V_1 & V_1' are finite unions of irred. varieties, the same holds for V.
- If not either V_1 or V_1' must be reducible. By symmetry, we may assume V_1 is red. & we write it as $V_1 = V_2 \cup V_2'$.

Continuing in this fashion, we are left with a chain

$$(*) \qquad V \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots\cdots$$

This infinite sequence never stabilizes if the statement fails for V. Taking $I(-)$ we get an ascending chain of ideals:

$$I(V) \subseteq I(V_1) \subseteq I(V_2) \subseteq \cdots\cdots$$

in the Noetherian ring $\mathbb{K}[x_1, \ldots, x_n]$. So this sequence must stabilize.

Thus $\exists N$ st $\forall k \geq N$ $\quad I(V_k) = I(V_{k+1})$

Taking $V(-)$ again & using Proposition 5, we conclude

$$V_k = V(I(V_k)) = V(I(V_{k+1})) = V_{k+1} \qquad \forall k \geq N$$

This contradicts our construction $(*)$ $\square$

Challenges: ① How to detect irreducibility?

② How to perform irreducible decompositions in practice?

For ① we have an easy characterization.

Proposition 1: A variety $W \subseteq \mathbb{A}^n$ is irreducible $\Leftrightarrow$ $I(W)$ is a prime ideal

Proof We prove both implications

($\Rightarrow$) Pick $f, g \in \mathbb{K}[\underline{x}]$ with $fg \in I(W)$.

Then $W = V(\langle I(W), f\rangle) \cup V(\langle I(W), g\rangle)$ will give a decomposition

If $f, g \notin I(W)$, this decomposition would be nontrivial, contradicting our irreducibility assumption on W.

($\Leftarrow$) We argue by contradiction & assume $W = W_1 \cup W_2$ is a nontrivial decomposition of W. In particular, we know that $W_1 \not\subseteq W_2 \not\subseteq W_1$. Equivalently, by Corollary 1 §3.2, this gives $\underset{\mathfrak{a}_1}{\underbrace{I(W_1)}} \not\subseteq \underset{\mathfrak{a}_2}{\underbrace{I(W_2)}} \not\subseteq I(W_1)$. So $\exists f \in \mathfrak{a}_1 \smallsetminus \mathfrak{a}_2$ & $g \in \mathfrak{a}_2 \smallsetminus \mathfrak{a}_1$.

By Lemma 2 §3.1, we know $I(W_1 \cup W_2) = \mathfrak{a}_1 \cap \mathfrak{a}_2 \supseteq \mathfrak{a}_1 \cdot \mathfrak{a}_2$

true for any pair of ideals

Then $fg \in \mathfrak{a}_1 \cdot \mathfrak{a}_2 \subseteq I(W)$ & $f, g \notin I(W)$ contradiction our assumption that $I(W)$ was prime. $\square$

For ② we will translate a decomposition of W into a decomposition of the (radical) ideal $I(W)$ as an intersection of prime ideals of $\mathbb{K}[x_1,\ldots,x_n]$

Next, we generalize this to arbitrary ideals of Noetherian commutative rings, replacing prime ideals by primary ones.

§2. Primary ideals

For today we fix R to be a Noetherian commutative ring (eg $\mathbb{K}[x_1,\ldots,x_n]$)

Definition: An ideal $q \subseteq R$ is called primary if it is proper and the following condition holds:

"If $a,b\in R$ satisfies $a\cdot b\in q$ & $a\notin q$ $\Longrightarrow$ $b^m\in q$ for some $m\geq 1$"
(equiv., $b\in\sqrt{q}$)

⚠ The definition is NOT symmetric in a & b.

Observation: Equivalently, every zero divisor of R/q is nilpotent (this one is symmetric!)
(see HW2)

Lemma 1: If $q\subseteq R$ is primary, then $\sqrt{q}$ is a prime ideal.

Proof Pick $a,b\in R$ with $a\cdot b\in \mathfrak{p}:=\sqrt{q}$. Then, $\exists\, m\geq 1$ with $(ab)^m=a^m b^m\in q$

By the definition of primary we have either $\underset{\substack{\Downarrow\\ a\in\mathfrak{p}}}{a^m\in q}$ or $b^m\in\sqrt{q}$, so $\underset{\substack{\Downarrow\\ b\in\mathfrak{p}}}{b\in\sqrt{q}}$ □

Observation: The difference between q & $\mathfrak{p}=\sqrt{q}$ is algebraic & highlights the difference between a fat point (a point with multiplicity) vs the point viewed as a set. This will be irrelevant for affine varieties (all our ideals will be radical) but it will play a role in scheme theory.

EXAMPLES: ① $R=\underset{\text{UFD}}{\mathbb{K}[x]}$ & $\mathfrak{a}=(x^n)$ Then, $\mathfrak{a}$ is primary

Pf/ If $ab\in(x^n)$ this forces $x^n\mid ab$ & x is an irreducible polynomial

So $a=x^k c$, $b=x^{n-k}d$ Thus, $a\in(x^n)$ or $b\in(x)=\sqrt{(x^n)}$.
$0\leq k\leq n$ $(k=0)$ $(k>0)$

Conclusion: If $a\notin(x^n)$ we have $b\in\sqrt{\mathfrak{a}}$ □

② $R = \mathbb{K}[x,y]$ & $\mathfrak{q} = (x, y^2)$. Then, $\mathfrak{q}$ is primary but not a power of a prime ideal.

Pf/ • Easy to see $\mathfrak{P} = (x,y) = \sqrt{\mathfrak{q}}$ is maximal $\Rightarrow$ prime

$$\mathfrak{P}^2 = (x^2, y^2, xy) \underset{x \notin \mathfrak{P}^2}{\subsetneq} \mathfrak{q} \underset{y \notin \mathfrak{q}}{\subsetneq} \mathfrak{P}.$$

If $\mathfrak{q} = \mathfrak{X}^m$ for some prime ideal $\mathfrak{X}$ then $\sqrt{\mathfrak{q}} = \sqrt{\mathfrak{X}^m} = \mathfrak{X}$ ie $\mathfrak{X} = \mathfrak{P}$.

• In what remains, we check that $\mathfrak{q}$ is primary. Pick $f, g \in \mathbb{K}[x,y]$ with $f \cdot g \in \mathfrak{q}$.

Write
$$f = a_0 + x\, f_1(x,y) + y\, f_2(y)$$
$$g = b_0 + x\, g_1(x,y) + y\, g_2(y)$$
with $a_0, b_0 \in \mathbb{K}$

Write $fg = a_0 b_0 + x\big(b_0 f_1(x,y) + a_0 g_1(x,y) + y(f_1(x,y) g_2(y) + g_1(x,y) f_2(y)) + f_1(x,y)\, g_1(x,y)\, x\big)$
$$+\, y\,(a_0 g_2(y) + b_0 f_2(y)) \in (x, y^2) \qquad (*)$$

Note: $f \notin \mathfrak{q}$ means $a_0 \neq 0$ or $(a_0 = 0$ & $f_2(0) \neq 0)$

Assuming $f \notin \mathfrak{q}$, we want to conclude that $g \in \sqrt{\mathfrak{q}} = (x,y)$ (ie $b_0 = 0$) We analyze 2 cases depending on the 2 options for f:

CASE 1: $a_0 \neq 0$

$fg \in \mathfrak{q} \overset{(*)}{\Longleftrightarrow} \overset{0}{\overset{\shortparallel}{a_0 b_0}} + y\,(a_0 g_2(y) + b_0 f_2(y)) \in (y^2) \subseteq \mathbb{K}[y]$

This forces $b_0 = 0$ (& $y \mid a_0 g_2(y) + b_0 f_2(y)$), so $g \in \sqrt{\mathfrak{q}}$ □

CASE 2: $a_0 = 0$ & $y \nmid f_2(y)$

$fg \in \mathfrak{q}$ simplifies to $fg = 0 + x(\quad\quad) + y\, b_0 f_2(y) \in (x, y^2)$

So $fg \in \mathfrak{q} \Leftrightarrow y \mid b_0 f_2(y)$ in $\mathbb{K}[y]$

Since $f_2(0) \neq 0$ by assumption, we conclude $b_0 = 0$ ie $g \in \sqrt{\mathfrak{q}}$ □

Note: In both examples we have $\sqrt{\mathfrak{q}}$ is a maximal ideal of R. This is a general fact.

Proposition 2: If R is a commutative Noetherian ring & $\mathfrak{q}$ is an ideal whose radical is maximal, then $\mathfrak{q}$ is primary.

Proof See HW2

EXAMPLE: ③ $R = \mathbb{K}[x,y,z]/(xy-z^2) \supseteq \mathfrak{p} = (\bar{x},\bar{z})$. We have that $\mathfrak{p}$ is a prime ideal but $\mathfrak{p}^2$ is not primary

Pf/ • To check $\mathfrak{p}$ is prime, we confirm $R/\mathfrak{p}$ is an integral domain.

$$R/\mathfrak{p} = \frac{\mathbb{K}[x,y,z]/(xy-z^2)}{(x,z,xy-z^2)/(xy-z^2)} \underset{\text{Iso Thm}}{\simeq} \frac{\mathbb{K}[x,y,z]}{(x,z,xy-z^2)} \simeq \frac{\mathbb{K}[x,y,z]}{(x,z)} \simeq \mathbb{K}[y]$$ which is a domain

• $\mathfrak{p}^2 = (\bar{x},\bar{z})^2 = (\bar{x}^2, \bar{z}^2, \bar{x}\bar{z}) = (\bar{x}^2, \bar{z}^2, \bar{x}\bar{z})$

BUT $\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{p}^2$ & $\underset{(1)}{\bar{x} \notin \mathfrak{p}^2}$ but $\underset{(2)}{\bar{y} \notin \sqrt{\mathfrak{p}^2}} = \mathfrak{p}$, so $\mathfrak{p}^2$ is not primary

↳ $\mathfrak{p}$ is prime (Lemma 2 below)

(1) & (2) can be checked by lifting to $\mathbb{K}[x,y,z]$

(1) $\bar{x} \notin \mathfrak{p}^2 \iff x \notin (x^2, z^2, xz, xy-z^2) = (x^2, xz, xy, z^2) \subseteq \mathbb{K}[x,y,z]$

(2) $\bar{y} \notin \sqrt{\mathfrak{p}^2} = \mathfrak{p} \iff y \notin \langle x, z, xy-z^2\rangle = \langle x, z\rangle \subseteq \mathbb{K}[x,y,z]$.

Remark In Ex ③ have $\bar{y} \notin \mathfrak{p}^2$ but $\bar{x} \in \sqrt{\mathfrak{p}^2}$, so this confirms the definition of Primary ideals is not symmetric in a & b

Lemma 2: (1) Prime ideals are radical

(2) For any prime ideal $\mathfrak{p}$ & any $n \geq 1$ we have: $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$

Proof: • (1) is an easy induction ($a^n = a(a^{n-1}) \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$)

• For (2), we show the double inclusion:

($\subseteq$) $\mathfrak{p}^n \subseteq \mathfrak{p} \Rightarrow \sqrt{\mathfrak{p}^n} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$

($\supseteq$) Given $a \in \mathfrak{p}$ we have $a^n \in \mathfrak{p}^n$ so $a \in \sqrt{\mathfrak{p}^n}$. Thus $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}^n}$. □

Summary of examples: • $\mathfrak{q}$ prime $\Rightarrow$ $\mathfrak{q}$ primary

• $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ = power of a prime ideal (Ex 2)

• $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ is primary (Ex 3)

• $\sqrt{\mathfrak{q}}$ is maximal $\Rightarrow$ $\mathfrak{q}$ is primary

GOAL: Write any ideal of R as a finite intersection of primary ideals with different radicals. (*) [Primary Decomposition of ideals of R]

For ideals $I(W)$ we'll write $\sqrt{I(W)} = I(W)$ as a finite intersection of prime ideals ($=\sqrt{\mathfrak{q}}$). We'll call them the associated primes of W (or $I(W)$). They come in 2 flavors: minimal or embedded. The latter ones come from embedded components of W. More on this on Lecture 5.

(*) Q: How can we ensure different primary ideals in the decomposition will have different radicals?
To show this we need one simple statement:

Proposition 3: If $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are primary ideals of R with $\sqrt{\mathfrak{q}_1} = \cdots = \sqrt{\mathfrak{q}_k} = \mathfrak{p}$ then $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ is also primary and $\sqrt{\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k} = \mathfrak{p}$.

Proof. The second statement follows directly from Lemma 3 (below).
- It remains to show that $\mathfrak{q} := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ is primary.

Fix $a, b \in R$ with $ab \in \mathfrak{q}$ & assume $a \notin \mathfrak{q}$. This means:
(i) $ab \in \mathfrak{q}_j \quad \forall\, j = 1, \ldots, k$
(ii) $\exists\, j_0$ with $a \notin \mathfrak{q}_{j_0}$

Since $ab \in \mathfrak{q}_{j_0}$, $a \notin \mathfrak{q}_{j_0}$ & $\mathfrak{q}_{j_0}$ is primary, we get $b \in \sqrt{\mathfrak{q}_{j_0}} = \mathfrak{p} = \sqrt{\mathfrak{q}}$ □

Lemma 3: For any pair of ideals $\mathfrak{a}_1$ & $\mathfrak{a}_2$ we have $\sqrt{\mathfrak{a}_1 \cap \mathfrak{a}_2} = \sqrt{\mathfrak{a}_1} \cap \sqrt{\mathfrak{a}_2}$

Pf/ ($\subseteq$) is clear because $\mathfrak{a}_1 \cap \mathfrak{a}_2 \subseteq \mathfrak{a}_i$ for $i = 1, 2$ & $\sqrt{}$ preserves inclusions.

($\supseteq$) If $a \in \sqrt{\mathfrak{a}_1} \cap \sqrt{\mathfrak{a}_2} \Rightarrow \exists\, N_1, N_2 \geq 1$ s.t. $a^{N_1} \in \mathfrak{a}_1$ & $a^{N_2} \in \mathfrak{a}_2$. Thus, for $N = \max\{N_1, N_2\}$ we have $a^N \in \mathfrak{a}_1 \cap \mathfrak{a}_2$, ie $a \in \sqrt{\mathfrak{a}_1 \cap \mathfrak{a}_2}$ □