Lecture V: Primary Decompositions & minimal primes of ideals

Fix R = Noetherian commutative ring

Recall: $\mathfrak{a} \subsetneq R$ ideal is primary if $ab \in \mathfrak{a}$ & $a \notin \mathfrak{a} \Rightarrow b \in \sqrt{\mathfrak{a}}$

Lemma: (1) $\mathfrak{q} \subseteq R$ primary $\Rightarrow \sqrt{\mathfrak{q}}$ prime.

(2) $\sqrt{\mathfrak{q}} \subset R$ is maximal $\Rightarrow \mathfrak{q}$ is primary

§1. More on irreducible ideals:

GOAL: Write any ideal of R as an intersection of finitely many primary ideals with different radicals. [Primary decompositions]

Proposition 1: If $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are primary ideals of R with $\sqrt{\mathfrak{q}_1} = \cdots = \sqrt{\mathfrak{q}_k} = \mathfrak{P}$ then $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ is also primary and $\sqrt{\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k} = \mathfrak{P}$.

Proof. The second statement follows directly from Lemma 1 (below).

- It remains to show that $\mathfrak{q} := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ is primary.

Fix $a, b \in R$ with $ab \in \mathfrak{q}$ & assume $a \notin \mathfrak{q}$. This means:

(i) $ab \in \mathfrak{q}_j \quad \forall\, j = 1, \ldots, k$

(ii) $\exists\, j_0$ with $a \notin \mathfrak{q}_{j_0}$

Since $ab \in \mathfrak{q}_{j_0}$, $a \notin \mathfrak{q}_{j_0}$ & $\mathfrak{q}_{j_0}$ is primary, we get $b \in \sqrt{\mathfrak{q}_{j_0}} = \mathfrak{P} = \sqrt{\mathfrak{q}}$ □

Lemma 1: For any pair of ideals $\mathfrak{a}_1$ & $\mathfrak{a}_2$ we have $\sqrt{\mathfrak{a}_1 \cap \mathfrak{a}_2} = \sqrt{\mathfrak{a}_1} \cap \sqrt{\mathfrak{a}_2}$

Pf/ ($\subseteq$) is clear because $\mathfrak{a}_1 \cap \mathfrak{a}_2 \subseteq \mathfrak{a}_i$ for $i = 1, 2$ & $\sqrt{\ }$ preserves inclusions.

($\supseteq$) If $a \in \sqrt{\mathfrak{a}_1} \cap \sqrt{\mathfrak{a}_2} \Rightarrow \exists\, N_1, N_2 \geq 1$ s.t. $a^{N_1} \in \mathfrak{a}_1$ & $a^{N_2} \in \mathfrak{a}_2$. Thus, for $N = \max\{N_1, N_2\}$ we have $a^N \in \mathfrak{a}_1 \cap \mathfrak{a}_2$, ie $a \in \sqrt{\mathfrak{a}_1 \cap \mathfrak{a}_2}$ □

- To build primary decompositions of ideals, we'll take a ditour. More precisely, we'll:

(1) define irreducible ideals.

(2) construct "irreducible decomp." of ideals (Noetherianness is key here!)

(3) show that irreducible ideals are primary.

§2 Irreducible ideals:

<u>Definition</u>: An ideal $\mathfrak{a} \subseteq R$ is called <u>irreducible</u> if the following holds:

"$\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$ for some ideals $\mathfrak{a}_1, \mathfrak{a}_2$ of R $\Longrightarrow$ $\mathfrak{a} = \mathfrak{a}_1$ or $\mathfrak{a} = \mathfrak{a}_2$"

<u>Note</u>: The terminology matches that of affine subvarieties of $\mathbb{A}^n$.

$$\left[\quad \mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2 \quad \Rightarrow \quad V(\mathfrak{a}) = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) \quad \right]$$

The next result gives us a large class of irreducible ideals:

<u>Lemma 2</u>: Every prime ideal is irreducible

<u>Proof</u>: Pick $\mathfrak{p}$ a prime ideal & write $\mathfrak{p} = \mathfrak{a}_1 \cap \mathfrak{a}_2$ for 2 ideals $\mathfrak{a}_1, \mathfrak{a}_2$

Therefore $\mathfrak{p} \subseteq \mathfrak{a}_1$ & $\mathfrak{p} \subseteq \mathfrak{a}_2$.

By construction $\mathfrak{a}_1 \cdot \mathfrak{a}_2 \subseteq \mathfrak{a}_1 \cap \mathfrak{a}_2 = \mathfrak{p}$. This gives:

<u>Claim</u>: $\mathfrak{a}_1 \subseteq \mathfrak{p}$ or $\mathfrak{a}_2 \subseteq \mathfrak{p}$.

Pf/ By contradiction, if $a_1 \in \mathfrak{a}_1 \setminus \mathfrak{p}$ & $a_2 \in \mathfrak{a}_2 \setminus \mathfrak{p}$, then $a_1 a_2 \notin \mathfrak{p}$ because $\mathfrak{p}$ is prime, but $a_1 \cdot a_2 \in \mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{p}$. Contradiction!

We conclude from here that $\mathfrak{a}_1 = \mathfrak{p}$ or $\mathfrak{a}_2 = \mathfrak{p}$. $\square$

⚠ primary $\not\Rightarrow$ irreducible (see HW2)

Our next results guarantees "irreducible decompositions" of ideals exist.

<u>Theorem 1</u>: Assume R is a Noetherian commutative ring. Then, any ideal of R is a finite intersection of irreducible ideals.

<u>Proof</u>: We argue by contradiction & assume the set:

$$\Sigma = \{ \mathfrak{a} \subseteq R \text{ ideal} : \mathfrak{a} \text{ is not a finite intersection of irred. ideals} \}$$

is non-empty. Since R is Noetherian, Σ must have a maximal element wrt inclusion. Say $\mathfrak{a} \in \Sigma$ is such a maxl element.

• Since $\mathfrak{a} \in \Sigma$, we know that $\mathfrak{a}$ cannot be irreducible, so we can write $\mathfrak{a}$ as $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ for two ideals $\mathfrak{b}, \mathfrak{c}$ with $\mathfrak{a} \subsetneq \mathfrak{b}$ & $\mathfrak{a} \subsetneq \mathfrak{c}$.

• Since $\mathfrak{a}$ was maximal in Σ, then $\mathfrak{b}, \mathfrak{c} \notin \Sigma$. Thus we can decompose $\mathfrak{b}$ & $\mathfrak{c}$:

$$\mathfrak{b} = \mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_s \quad \text{with } \mathfrak{b}_i \text{ irred. } \forall i = 1, \dots, r$$

$$\mathfrak{c} = \mathfrak{c}_1 \cap \dots \cap \mathfrak{c}_r \quad \text{— } \mathfrak{c}_j \text{ — } \forall j = 1, \dots, s$$

This gives a decomposition of $\mathfrak{a} = \mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_s \cap \mathfrak{c}_1 \cap \dots \cap \mathfrak{c}_s$, so $\mathfrak{a} \notin \Sigma$.

Contradiction! <u>Conclusion</u>: $\Sigma = \emptyset$, as we wanted to show. □

To finish, we need to show the validity of (3):

<u>Lemma 3</u>: Irreducible ideals are primary ideals if R is Noetherian.

<u>Proof</u>: Fix $\mathfrak{a} \subseteq R$ irreducible ideal $\Rightarrow (0) \subseteq R/\mathfrak{a} = \tilde{R}$ is also an irred. ideal in the Noetherian commutative ring $\tilde{R}$. In addition: $\mathfrak{a} \subseteq R$ is primary iff $(0) \subseteq \tilde{R}$ is primary. Thus, it is enough to prove the statement for (0) when it's irreducible.

• Pick $a, b \in R$ & $ab = 0$. Assuming $a \neq 0$, <u>we want to show</u> $b^n = 0$ for some n.

For this, we consider the chain of ideals:

$$(**) \quad \mathrm{Ann}(b) \subseteq \mathrm{Ann}(b^2) \subseteq \dots \subseteq \mathrm{Ann}(b^k) \subseteq \mathrm{Ann}(b^{k+1}) \subseteq \dots$$

[<u>Recall</u> $\mathrm{Ann}(x) = \{y \in R : y \cdot x = 0\} \subseteq R$ is an ideal]

• Since R is Noetherian, the ascending chain $(**)$ stabilizes, so $\exists n > 0$ st

$$\mathrm{Ann}(b^n) = \mathrm{Ann}(b^k) \quad \forall k \geq n.$$

<u>Claim</u>: $(0) = (b^n) \cap (a)$

Pf/ We prove the inclusion $(\supseteq)$. Pick $x \in (b^n) \cap (a)$.

$$\left.\begin{array}{ll} (1) \bullet\ x \in (a) \Rightarrow & xb \in (ab) = (0) \\ (2) \bullet\ x \in (b^n) \Rightarrow & x = b^n y \text{ for some } y \in R \end{array}\right\} \Rightarrow \begin{array}{l} b^{n+1} y = 0, \text{ ie} \\ y \in \mathrm{Ann}(b^{n+1}) = \mathrm{Ann}(b^n) \end{array}$$

Plugging this back in (2) gives $x = b^n y = 0$. □

• Since (0) is irreducible by assumption & $(a) \neq (0)$ by hypothesis, we conclude that $(0) = (b^n)$ ie $x = 0$ as we wanted to show. $\square$

Corollary 1: We can decompose any ideal as a finite intersection of primary ideals with different radicals.

Pf/ Combine Theorem 1 & Lemma 3 to build the decomposition. Group together those primary ideals with the same radical via Proposition 1 to get the second part of the statement.

For radical ideals we get:

Corollary 2: Every radical ideal is a finite intersection of prime ideals

Remark: The construction for $K[x_1, \dots x_n]$ & power series rings is due to Lasker (1905). It uses induction on n & complicated eliminations. The statement and proof for any Noetherian comm. ring discussed above is much cleaner. It is due to E Noether (1921).

Natural Question: (1) Are these decompositions unique?
(2) What do the radicals of the primary components have to do with $\mathfrak{a}$?

§2 Minimal Primary Decompositions & minimal primes

Next goal: ① characterize primary ideals in the decomp
② analyze possible uniqueness.

Theorem 2: Let R be a Noetherian commutative ring & $\mathfrak{a} \subseteq R$ a proper ideal. Then $\exists\, \mathfrak{q}_1, \dots, \mathfrak{q}_r$ primary ideals (so proper) such that

(1) $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$ (primary decomp)

(2) $\mathfrak{p}_1 = \sqrt{\mathfrak{q}_1}, \dots, \mathfrak{p}_r = \sqrt{\mathfrak{q}_r}$ are all distinct primes.

(3) [Minimality] The intersection in (1) has no irrelevant terms., that is $\forall\, j = 1, \dots, r$ $\mathfrak{q}_j \not\supset \bigcap_{i \neq j} \mathfrak{q}_i$ (no $\mathfrak{q}_j$ can be omitted from (1))

Name: $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ are called primary components of $\mathfrak{a}$ for this decomposition.

Proof: (1) follows from any irreducible decomp of $\mathfrak{a}$ (Theorem 1 § 3.2)

(2) comes from grouping together primary ideals featured in (1) with the same radical (Proposition 1 § 4.1).

(3) is obtained from (1) by removing redundant $\mathfrak{q}_{ij}$'s on (RHS)

Definition: The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is called the set of Associated primes of $\mathfrak{a}$. We denote it by $\boxed{\mathrm{Assoc}(\mathfrak{a})}$

Remark: The construction of $\mathrm{Assoc}(\mathfrak{a})$ is independent of the minimal decomp, but this is NOT obvious! We'll see it in § 5.2

One thing we can show is the minimal primes over $\mathfrak{a}$ always lie in $\mathrm{Assoc}(\mathfrak{a})$.

Definition: Given $\mathfrak{a}$ ideal & $\mathfrak{p}$ prime ideal with $\mathfrak{a} \subseteq \mathfrak{p}$, we say is a minimal prime of $\mathfrak{a}$ if $\mathfrak{p} = \mathfrak{a}$ or if $\mathfrak{a}$ is NOT prime & $\nexists\, \mathfrak{p}'$ prime with $\mathfrak{a} \subsetneq \mathfrak{p}' \subsetneq \mathfrak{p}$. We write $\boxed{\mathrm{Min}(\mathfrak{a})}$ for the set of minimal primes of $\mathfrak{a}$.

- Our next result ensures $\mathrm{Min}(\mathfrak{a})$ is finite.

Lemma 4: For any proper ideal $\mathfrak{a}$ of a Noetherian commutative ring R we have

$$\mathrm{Min}(\mathfrak{a}) \subseteq \mathrm{Assoc}(\mathfrak{a})$$

Proof: Next Time.

Corollary 3: The set $\mathrm{Min}(\mathfrak{a})$ is finite

Note: We are not using that $\mathrm{Assoc}(\mathfrak{a})$ is independent of the minimal primary decomposition of $\mathfrak{a}$, just that it is finite for any such decomposition.