Lecture VI: Associated primes of ideals, uniqueness of minimal primary comp.

Fix R = Noetherian commutative ring

Theorem: Let R be a Noetherian commutative ring & $\mathfrak{a} \subseteq R$ a proper ideal. Then $\exists\, \mathfrak{q}_1, \ldots, \mathfrak{q}_r$ primary ideals (so proper) such that

(1) $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ (primary decomp)

(2) $\mathfrak{p}_1 = \sqrt{\mathfrak{q}_1}, \ldots, \mathfrak{p}_r = \sqrt{\mathfrak{q}_r}$ are all distinct primes.

(3) [Minimality] The intersection in (1) has no irrelevant terms, that if $\forall j = 1, \ldots, r$ $\mathfrak{q}_j \not\supseteq \bigcap_{i \neq j} \mathfrak{q}_i$ (no $\mathfrak{q}_j$ can be omitted from (1))

Name: $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are called primary components of $\mathfrak{a}$ for this decomposition.

Next goal: ① characterize primary ideals in the minimal decomposition

② analyze possible uniqueness on either $\mathfrak{q}_i$'s or $\mathfrak{p}_i$'s.

Special case: $\mathfrak{a}$ prime. Then $\underset{\text{prime}}{\mathfrak{a}} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ gives $\mathfrak{q}_i \subseteq \mathfrak{a}$ for some i by prime avoidance. Since $\mathfrak{a} \subseteq \mathfrak{q}_i$ we get $\mathfrak{a} = \mathfrak{q}_i$.

So a minimal primary decomposition has $r = 1$.

§1. Associated primes & minimal primes:

Definition: The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is called the set of Associated primes of $\mathfrak{a}$. We denote it by $\boxed{\mathrm{Assoc}(\mathfrak{a})}$

Remark: The construction of $\mathrm{Assoc}(\mathfrak{a})$ is independent of the minimal decomp, but this is NOT obvious! We'll see it in § 5.2

One thing we can show is the minimal primes over $\mathfrak{a}$ always lie in $\mathrm{Assoc}(\mathfrak{a})$.

Definition: Given $\mathfrak{a}$ ideal & $\mathfrak{p}$ prime ideal with $\mathfrak{a} \subseteq \mathfrak{p}$, we say is a minimal prime of $\mathfrak{a}$ if $\mathfrak{p} = \mathfrak{a}$, or if $\mathfrak{a}$ is NOT prime & $\nexists\, \mathfrak{p}'$ prime with $\mathfrak{a} \subsetneq \mathfrak{p}' \subsetneq \mathfrak{p}$. We write $\boxed{\mathrm{Min}(\mathfrak{a})}$ for the set of minimal primes of $\mathfrak{a}$.

- Our next result ensures $\mathrm{Min}(\mathfrak{a})$ is finite.

Proposition 1: For any proper ideal $\mathfrak{a}$ of a Noetherian commutative ring R we have
$$\mathrm{Min}(\mathfrak{a}) \subseteq \mathrm{Assoc}(\mathfrak{a})$$

Proof: Write a minimal primary decomposition of $\mathfrak{a}$ (Theorem 2 §5.2)
$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \subseteq \underset{\text{prime}}{\mathfrak{p}}$$

- **Prime avoidance** says $\exists\, i = 1, \ldots, r$ with $\mathfrak{q}_i \subseteq \mathfrak{p}$ (*)

Recall: Induction on r reduces Prime Avoidance to the following statement:
$$\text{“} \mathfrak{b} \cap \mathfrak{c} \subseteq \mathfrak{p},\ \mathfrak{b}\ \&\ \mathfrak{c} \text{ ideals} \Longrightarrow \mathfrak{b} \subseteq \mathfrak{p} \text{ or } \mathfrak{c} \subseteq \mathfrak{p} \text{”}$$
Pf/ By contradiction. Assume $\exists\, b \in \mathfrak{b} \setminus \mathfrak{p}$, $c \in \mathfrak{c} \setminus \mathfrak{p}$. Then, $b.c \in \mathfrak{b}.\mathfrak{c} \subseteq \mathfrak{b} \cap \mathfrak{c} \subseteq \mathfrak{p}$ but $\mathfrak{p}$ is prime & $b, c \notin \mathfrak{p}$ Contradiction!

- Taking radicals in (*) gives $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$ & $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}} \subseteq \mathfrak{p}_i \subseteq \mathfrak{p}$

Since $\mathfrak{p}$ is prime & minimal over $\mathfrak{a}$, then one of the following holds:

(1) $\mathfrak{p} = \mathfrak{a}$, or (2) $\mathfrak{a}$ is not prime & so $\mathfrak{p}_i = \mathfrak{p}$.

- If (1) holds, then $\underset{\text{prime}}{\mathfrak{a} = \sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r} \underset{\supseteq}{=} \sqrt{\mathfrak{q}_1} \cap \cdots \cap \sqrt{\mathfrak{q}_r}$ so

$\mathfrak{a} = \mathfrak{p} \supseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. By prime avoidance $\exists\, j = 1, \ldots, r$ with $\mathfrak{p}_j \subseteq \mathfrak{a}$

Since $\mathfrak{a} \subseteq \mathfrak{q}_j \subseteq \mathfrak{p}_j$ by construction, we get $\mathfrak{a} = \mathfrak{p} = \mathfrak{p}_j \in \mathrm{Assoc}(\mathfrak{a})$.

Thus, from (1) or (2), we get $\mathfrak{p} \in \mathrm{Assoc}(\mathfrak{a})$ □ □

Corollary 1: The minimal primes of $\mathfrak{a}$ appear on any minimal primary decomposition as radicals of some primary components

Definition: The set $\mathrm{Assoc}(\mathfrak{a}) \setminus \mathrm{Min}(\mathfrak{a})$ is called the set of <u>embedded primes</u> of $\mathfrak{a}$
(They correspond to "embedded components" of affine varieties)

§2 Uniqueness of minimal primary components:

Definition: We say a primary component $\mathfrak{q}$ of $\mathfrak{a}$ is minimal if $\sqrt{\mathfrak{q}} \in \operatorname{Min}(\mathfrak{a})$.

Theorem 1: Fix an ideal $\mathfrak{a}$ of a Noetherian commutative ring R & a minimal primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. If $\sqrt{\mathfrak{q}_j} \in \operatorname{Min}(\mathfrak{a})$, then $\mathfrak{q}_j$ is uniquely determined by $\mathfrak{a}$ (& thus features in any minimal primary decomp of $\mathfrak{a}$).

Our next result will be useful to prove Theorem 1.

Lemma 1: Fix $\mathfrak{q}$, $\mathfrak{q}'$ 2 primary components of an ideal $\mathfrak{a}$ (so $\mathfrak{p} := \sqrt{\mathfrak{q}} \neq \sqrt{\mathfrak{q}'}$)
If $\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})$, then $\mathfrak{q}' \not\subseteq \mathfrak{p}$.

Proof: By contradiction: If $\mathfrak{q}' \subseteq \mathfrak{p} \Rightarrow \mathfrak{a} \subseteq \mathfrak{q}' \subseteq \underbrace{\sqrt{\mathfrak{q}'}}_{\text{prime}} \subseteq \sqrt{\mathfrak{p}} = \underbrace{\mathfrak{p}}_{\text{minimal}}$, so □

(1) $\mathfrak{a}$ is prime so $\mathfrak{q} = \mathfrak{q}' = \mathfrak{a}$, which cannot happen,

or

(2) $\mathfrak{a}$ is not prime & $\sqrt{\mathfrak{q}'} = \mathfrak{p} = \sqrt{\mathfrak{q}}$ Contradiction!

⚠ The statement can fail if $\mathfrak{p} \notin \operatorname{Min}(\mathfrak{a})$ (see HW2 Problem 8)

Proof of Theorem 1: Write $\mathfrak{p}_\ell = \sqrt{\mathfrak{q}_\ell}$ $\forall \ell$.

Assume $\operatorname{Min}(\mathfrak{a}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ (if not, reorder the $\mathfrak{q}$'s)

Given $i = 1, \ldots, k$ we want to give a characterization of $\mathfrak{q}_i$ in terms of $\mathfrak{a}$ & $\mathfrak{p}_i$.

We will need to do some localization away from $\mathfrak{p}_i$.

We simplify notation & write $\mathfrak{q} := \mathfrak{q}_i$ & $\mathfrak{p} := \mathfrak{p}_i$.

- Since $\mathfrak{p}$ is prime, $S = R \setminus \mathfrak{p}$ is a multiplicatively closed set in R $\left(\begin{array}{l} \cdot 1 \in S \\ \cdot a, b \in S \Rightarrow ab \in S \end{array}\right)$

$\Rightarrow$ We can consider the localization $S^{-1}R = \{\frac{r}{s} : r \in R, s \in S\}/\sim$

Here: $\frac{a}{s} \sim \frac{a'}{s'}$ if $\exists s'' \in S$ with $s''(s'a - sa') = 0$.

- $S^{-1}R$ is a ring with operations modeled on those in $\mathbb{Q}$.

- We have a ring homomorphism $j: R \longrightarrow S^{-1}R$ (the localization).
$r \longmapsto \frac{r}{1}$

Lemma 2 below gives the desired characterization for $\mathfrak{q}$ as $j^*(S^{-1}\mathfrak{a})$. $\square$

Lemma 2: For $\mathfrak{p}$, $\mathfrak{q}$ & $\mathfrak{a}$ as above we have $\mathfrak{q} = j^*(S^{-1}\mathfrak{a})$

Proof By construction, $j^*(S^{-1}\mathfrak{a}) = \{r \in R : sr \in \mathfrak{a} \text{ for some } s \in S\}$

We prove the statement by a double inclusion:

($\supseteq$) Pick $a \in j^*(S^{-1}\mathfrak{a}) \subseteq j^*(S^{-1}\mathfrak{q}) \Rightarrow \exists\, t \in S$ such that $at = ta \in \mathfrak{q}$.
$\mathfrak{a} \subseteq \mathfrak{q}$

Since $\mathfrak{q}$ is primary this gives either $a \in \mathfrak{q}$ or $t \in \sqrt{\mathfrak{q}} = \mathfrak{p}$.

But $t \in S$ means $t \notin \mathfrak{p}$, so this forces $a \in \mathfrak{q}$ ✓

($\subseteq$) By Lemma 2 we know that for any $\mathfrak{q}_t \neq \mathfrak{q}$ we have $\mathfrak{q}_t \not\subseteq \mathfrak{p}$. In particular: $S^{-1}\mathfrak{q}_t = S^{-1}R$

As a consequence: $S^{-1}\mathfrak{a} = S^{-1}(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r) \overset{\text{Exercise}}{=} S^{-1}\mathfrak{q}_1 \cap \cdots \cap S^{-1}\mathfrak{q}_r$

$= (S^{-1}R) \cap \cdots \cap \underbrace{S^{-1}\mathfrak{q}}_{i^{th}\text{ spot}} \cap \cdots \cap S^{-1}R = S^{-1}\mathfrak{q}$.

$\Rightarrow j^*(S^{-1}\mathfrak{a}) = j^*(S^{-1}\mathfrak{q}) \supseteq \mathfrak{q}$. $\square$
↳ always true!

⚠ The proof of Lemma 2 fails if $\mathfrak{p} \notin \operatorname{Min}(\mathfrak{a})$ because Lemma 1 can fail.

§3. Associated Primes:

Theorem 2: $\operatorname{Assoc}(\mathfrak{a})$ is independent of any choice of minimal primary decomposition of the ideal $\mathfrak{a}$.

To prove the statement we need the following auxiliary result:

Lemma 3: Fix a commutative ring R & a primary ideal $\mathfrak{q}$ in R. Write $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

For $x \in R$ we have:

(1) $x \in \mathfrak{q} \iff (\mathfrak{q} : x) = R$

(2) $x \notin \mathfrak{q} \Rightarrow (\mathfrak{q} : x)$ is primary & $\sqrt{(\mathfrak{q}:x)} = \mathfrak{p}$.

(3) $x \notin \mathfrak{p} \Rightarrow (\mathfrak{q} : x) = \mathfrak{q}$.

Here $(\mathfrak{q} : x) = \{ a \in R : ax \in \mathfrak{q} \}$

Proof: (1) is by definition of $(\mathfrak{q}:x)$ because $1 \in (\mathfrak{q}:x) \iff x = 1 \cdot x \in \mathfrak{q}$.

(3) We prove $(\subseteq)$ since $\mathfrak{q} \subseteq (\mathfrak{q}:x)$ is always valid.

Pick $a \in (\mathfrak{q}:x)$ ie $ax \in \mathfrak{q}$. Since $\mathfrak{q}$ is primary & $x \notin \sqrt{\mathfrak{q}}$ we have $a \in \mathfrak{q}$.

(2) We first show that $\sqrt{(\mathfrak{q}:x)} = \mathfrak{p}$ by double inclusion:

$(\supseteq)$ $\mathfrak{q} \subseteq (\mathfrak{q}:x)$ so $\sqrt{\mathfrak{q}} = \mathfrak{p} \subseteq \sqrt{(\mathfrak{q}:x)}$.

$(\subseteq)$ Pick $y \in (\mathfrak{q}:x)$ so $xy \in \mathfrak{q}$. Since $x \notin \mathfrak{q}$ & $\mathfrak{q}$ is primary, we have $y \in \sqrt{\mathfrak{q}} = \mathfrak{p}$.

Conclude $(\mathfrak{q}:x) \subseteq \mathfrak{p}$.

Next, we check $(\mathfrak{q}:x)$ is primary. By (1) we know it is a proper ideal of R.

Pick $a, b \in R$ with $a \notin (\mathfrak{q}:x)$ & $ab \in (\mathfrak{q}:x)$. Then, $ax \notin \mathfrak{q}$ but $axb \in \mathfrak{q}$

Since $\mathfrak{q}$ is primary, we get $b \in \sqrt{\mathfrak{q}} = \mathfrak{p} = \sqrt{(\mathfrak{q}:x)}$ as we wanted. $\square$

- The proof of Theorem 2 is a direct consequence of the following characterization of $\mathrm{Ass}(\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$

Proposition 2: Given a minimal primary decomp of $\mathfrak{a}$ with assoc. primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$

we have
$$\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} = \{ \sqrt{(\mathfrak{a}:x)} : x \in R \text{ \& } \sqrt{(\mathfrak{a}:x)} \text{ is a prime ideal} \}$$

depends on the min primary decomposition

independent of the min primary decomposition

Proof: Write a minimal primary decomposition of $\mathfrak{a}$

$$\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r \quad \text{with } \sqrt{\mathfrak{q}_i} = \mathfrak{p}_i.$$

Note that $\sqrt{(\mathfrak{a}:x)} = \sqrt{(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r : x)} \underset{\text{easy exercise}}{=} \bigcap_{j=1}^{r} \sqrt{(\mathfrak{q}_j : x)}$ (*)

By Lemma 3 we have 2 options for each $\sqrt{(\mathfrak{q}_j : x)} = \begin{cases} 1 & \text{if } x \in \mathfrak{q}_j \\ \mathfrak{p}_j & \text{else} \end{cases}$

Writing the non-trivial terms in (RHS) of (*) we get $\sqrt{(\mathfrak{a}:x)} = \bigcap_{j=1}^{k} \mathfrak{p}_{i_j}$

Next, we show the double inclusion of the sets in the statement:

($\supseteq$) If $\mathfrak{p} = \sqrt{(\mathfrak{a}:x)}$ is prime & $\underset{\text{Prime}}{\mathfrak{p}} = \bigcap_{j=1}^{k} \mathfrak{p}_{i_j}$. Prime avoidance says $\mathfrak{p} \supseteq \mathfrak{p}_{i_j}$ for some j

Since $\bigcap_{j=1}^{k} \mathfrak{p}_{i_j} \subseteq \mathfrak{p}_{i_j}$ we get $\mathfrak{p} = \mathfrak{p}_{i_j}$ so it lies in the (LHS) of (*)

($\subseteq$) Pick $\mathfrak{p} = \mathfrak{p}_j$ then as $\mathfrak{q}_j \not\supset \bigcap_{i \neq j} \mathfrak{q}_i$ (by the minimality of the decomp.)

we can pick $x \in \bigcap_{i \neq j} \mathfrak{q}_i \setminus \mathfrak{q}_j$ Then:

$$\sqrt{(\mathfrak{a}:x)} = \bigcap_{i=1}^{r} \sqrt{(\mathfrak{q}_i : x)} = \sqrt{(\mathfrak{q}_j : x)} \cap \bigcap_{i \neq j} \underbrace{\sqrt{(\mathfrak{q}_i : x)}}_{= R \text{ by Lemma 3(1) since } x \in \mathfrak{q}_i \text{ for } i \neq j} = \sqrt{(\mathfrak{q}_j : x)} \overset{x \notin \mathfrak{q}_j + \text{Lemma 3(2)}}{=} \mathfrak{p}_j$$

Conclusion: $\mathfrak{p}_j$ has the desired form for this choice of x, so $\mathfrak{p}_j$ is in the (RHS). □

In HW2, we'll have examples of primary decompositions.