Lecture X: Coordinate Rings & Morphisms

So far we have described the objects of interest (affine subvarieties of $\mathbb{A}^n_{\mathbb{K}}$)
Next, we construct functions between these objects to obtain a category.

§1. Coordinate Rings:

Definition: Given $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ we define its coordinate ring or the ring of polynomial functions on W as $\mathbb{K}[W] := \mathbb{K}[x_1, \dots, x_n]/_{I(W)}$

Note: We know from Lemma 4 §3.2 that $I(W)$ is radical, so $\mathbb{K}[W]$ has no nilpotents.
This lack of nilpotents will be dropped when dealing with Schemes.

Q: Why is this called the ring of polynomial functions?

A: A polynomial $f \in \mathbb{K}[x_1, \dots x_n]$ defines a function

$$f : \mathbb{A}^n_{\mathbb{K}} \longrightarrow \mathbb{K} = \mathbb{A}^1_{\mathbb{K}}\,.$$
$$\underline{p} \longmapsto f(\underline{p})$$

However, 2 functions f, g on $\mathbb{A}^n_{\mathbb{K}}$ restrict to the same function on $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ subvariety if, and only if, $f(\underline{p}) - g(\underline{p}) = 0 \quad \forall \underline{p} \in W$. Equivalently, iff $f - g \in I(W)$. Thus,
$\bar{f} = \bar{g}$ in $\mathbb{K}[V]$

- The lack of nilpotents for $\mathbb{K}[V]$ ensures that we don't have f with $f^m(\underline{p}) = 0 \ \forall \underline{p} \in V$ but $f \not\equiv 0$ on V.

⚠ The ring $\mathbb{K}[W]$ admits many presentations, so $I(W)$ cannot be recovered uniquely from $\mathbb{K}[W]$. (Eg $\mathbb{K}[\mathbb{A}^1_{\mathbb{K}}] = \mathbb{K}[x] = \mathbb{K}[x,y]/_{(y)}$)

- For affine schemes, the space is determined by its ring of functions, so the presentation will be irrelevant

Remark: W is irreducible $\iff$ $\mathbb{K}[W]$ is an integral domain.

- This construction allows us to build relative versions of $V(\cdot)$ & $I(\cdot)$ to free ourselves from the ambient variety $\mathbb{A}^n_{\mathbb{K}}$.

Definition: Fix $Y \subseteq \mathbb{A}^n_{\mathbb{K}}$ an affine subvariety.

(1) Given a subset $S \subseteq \mathbb{K}[Y]$ (polynomial functions on Y) we define its zero locus as $V_Y(S) = \{x \in Y : f(x) = 0 \ \forall f \in S\} \subset Y$.

Any such $V(S)$ is called an affine subvariety of Y.

(2) Given a subset $W \subset Y$, the ideal of W on Y is defined as

$$I_Y(W) = \{f \in \mathbb{K}[Y] : f(x) = 0 \ \forall x \in W\} \subseteq \mathbb{K}[Y]$$

The following results extend those when $Y = \mathbb{A}^n_{\mathbb{K}}$ to the relative setting:

Proposition 1: Let $Y \subseteq \mathbb{A}^n_{\mathbb{K}}$ be an affine variety. Then

(1) $V_Y(I_Y(W)) = W$ for all $W \subseteq Y$ affine variety.

(2) $I_Y(W) \subseteq \mathbb{K}[Y]$ is a radical ideal

(3) $I_Y(V_Y(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}$ for every ideal $\mathfrak{a}$ of $\mathbb{K}[Y]$

Relative Nullstellensatz Fix $Y \subseteq \mathbb{A}^n_{\mathbb{K}}$ an affine variety & assume $\overline{\mathbb{K}} = \mathbb{K}$

Given $\mathfrak{a} \subseteq \mathbb{K}[Y]$ ideal, we have

- Strong version: $I_Y(V_Y(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ in $\mathbb{K}[Y]$
- Weak version: $V_Y(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = (1) = \mathbb{K}[Y]$.

§2. Polynomial maps between affine varieties:

Given $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ & $W \subseteq \mathbb{A}^m_{\mathbb{K}}$ our next goal is to define maps $\varphi: V \longrightarrow W$ compatible with the algebra of affine varieties. What conditions we want to impose on φ?

(1) Maps should be continuous with respect to the Zariski Topology on both V & W

(2) If $W = \mathbb{A}^1_{\mathbb{K}}$ we should recover $\mathbb{K}[V]$ which we identified with polynomial maps $V \longrightarrow \mathbb{A}^1_{\mathbb{K}}$.

In general, maps $V \longrightarrow \mathbb{A}^m_{\mathbb{K}}$ built out of functions on V (ie $\mathbb{K}[V]$), so we'll need m-tuples of elements on $\mathbb{K}[V]$ to define φ. In short: $\varphi \longleftrightarrow \mathbb{K}[V]^m$.

(Reason: we have n natural projections $\pi_j : \mathbb{A}^n_{\mathbb{K}} \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ $\quad \underline{x} \longmapsto x_j \quad j=1,\dots,n$
& $f_j := \pi_j \circ inc \circ \varphi : V \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ should be a polynomial function on V.)

(3) Compositions should be allowed & $id : V \longrightarrow V$ should be allowed.

(4) Viewing $\mathbb{K}[W]$ as maps $W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ & $\mathbb{K}[V]$ as maps $V \longrightarrow \mathbb{A}^1_{\mathbb{K}}$
we see that any $g \in \mathbb{K}[W]$ should give us an element of $\mathbb{K}[V]$ via

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[rr, bend right, "g\circ\varphi"'] & W \arrow[r, "g"] & \mathbb{A}^1_{\mathbb{K}} \end{tikzcd}$$

This will lead to a map $\varphi^* : \mathbb{K}[W] \longrightarrow \mathbb{K}[V]$ called the pullback

In general, viewing $V \longrightarrow W \hookrightarrow \mathbb{A}^n_{\mathbb{K}}$ we see that maps
$\varphi : V \longrightarrow W$ come from polynomial maps whose image lies in W.

Definition: A polynomial map or morphism from V to W is a map
$\varphi : V \longrightarrow W$ such that there exists $f_1, \dots, f_m \in \mathbb{K}[V]$ with

$$\varphi(\underline{p}) = (f_1(\underline{p}), \dots, f_m(\underline{p})) \quad \forall \underline{p} \in V$$

The set of morphisms is denoted by $Hom(V, W)$.

Examples: (1) $Hom(\mathbb{A}^n_{\mathbb{K}}, \mathbb{A}^m_{\mathbb{K}}) = \mathbb{K}[x_1, \dots x_n]^m$

(2) $Hom(V, \mathbb{A}^1_{\mathbb{K}}) = \mathbb{K}[V]$

(3) If $V \subseteq W$ is subvariety, then $inc : V \hookrightarrow W$ lies in $Hom(V,W)$

($W \subseteq \mathbb{A}^m_{\mathbb{K}}$ so $\varphi = (\bar{x}_1, \dots, \bar{x}_m)$ $\quad \bar{x}_i \in \mathbb{K}[V] = \dfrac{\mathbb{C}[x_1 \cdots x_m]}{I(V)}$

In particular, $\mathbb{1}_V : V \longrightarrow V \in Hom(V, V)$.

Next, we check our wishlist of properties for $Hom(V,W)$:

Proposition 2: Any $\varphi \in Hom(V,W)$ is continuous when V & W are endowed with their respective Zariski Topologies.

Proof: If $Z \subseteq W$ is closed, then $\varphi^{-1}(Z) = \{p \in V : f_1(p), \dots, f_m(p) \in Z\}$

Since W is closed in $\mathbb{A}^m_{\mathbb{K}}$, then Z is closed in $\mathbb{A}^m_{\mathbb{K}}$ ie $Z = V(S)$ for some $S = \{g_1, \dots, g_r\} \subseteq \mathbb{K}[x_1, \dots, x_m]$. In particular,

$$\varphi^{-1}(Z) = V \cap V(\underbrace{g_1(f_1, \dots f_m)}_{\in \mathbb{K}[x_1,\dots,x_n]}, \dots, \underbrace{g_r(f_1, \dots, f_m)}_{\in \mathbb{K}[x_1,\dots,x_n]})$$

So, $\varphi^{-1}(Z)$ is an affine subvariety of $\mathbb{A}^n_{\mathbb{K}}$. & $\varphi^{-1}(Z) \subseteq V$. Thus, it is closed in the Zariski Topology of V.

Conclusion: preimages of closed sets of W are closed in V, so φ is continuous. $\square$

Next, we check conditions that maps on a category must satisfy:

Proposition 3: (1) If $\varphi \in Hom(V,W)$ & $\psi \in Hom(W,Z)$ then $\psi \circ \varphi \in Hom(V,Z)$.

(2) $\mathbb{1} = V \longrightarrow V \in Hom(V,W)$

Proof: (1) Compositions of polynomial functions are polynomial functions.

If $\varphi = (f_1 \dots, f_m)$ $\psi = (g_1, \dots, g_s)$ $g_i \in \mathbb{K}[W]$, $f_j \in \mathbb{K}[V]$.

Then $g_i(f_1, \dots, f_m)(p)$ is well-defined on $p \in V$ because $f_1(p), \dots, f_m(p) \in W$, ie the value is independent on the representative of $g_i \in \mathbb{K}[W]$ nor $f_j \in \mathbb{K}[V]$.

(2) $\mathbb{1} = (x_1, \dots, x_n)$ if $V \subseteq \mathbb{A}^n_{\mathbb{K}}$. $\square$

Our main theorem confirms that the data of a morphism is purely ring theoretic.

Definition: Given $\varphi \in Hom(V,W)$ & $g \in \mathbb{K}[W]$ we define the pullback $\varphi^* g = g \circ \varphi \in \mathbb{K}[V]$

Lemma 2: • $\varphi^* : \mathbb{K}[W] \longrightarrow \mathbb{K}[V]$ is a $\mathbb{K}$-algebra homomorphism

Proof: • We check the 3 conditions for being a ring homomorphism.

(i) $\varphi^*(g \pm f) = (g \pm f) \circ \varphi = g \circ \varphi \pm f \circ \varphi = \varphi^*(g) \pm \varphi^*(f)$.

(ii) $\varphi^*(1) = 1\circ\varphi = 1$

(iii) $\varphi^*(gf) = (gf)\circ\varphi = (g\circ\varphi)\cdot(f\circ\varphi) = \varphi^*(g)\cdot\varphi^*(f)$.

- Next, we check that $\varphi^*|_{\mathbb{K}} = inc_{\mathbb{K},\mathbb{K}[V]}$

$$\varphi^*(\bar{k}) = \bar{k}\circ\varphi = \bar{k} = inc_{\mathbb{K},\mathbb{K}[V]}(k). \qquad \square$$

Theorem 2: If $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ & $W \subseteq \mathbb{A}^m_{\mathbb{K}}$ are affine subvarieties, then $\varphi \mapsto \varphi^*$ defines a bijection:

$$\begin{array}{ccc} Hom(V,W) & \xrightarrow{\Phi} & \{\psi: \mathbb{K}[W]\to\mathbb{K}[V]\ \mathbb{K}\text{-algebra homomorphisms}\} \\ \varphi & \longmapsto & \varphi^* \end{array}$$

Proof: We consider the coordinate functions $y_1,\dots,y_m \in \mathbb{K}[W]$

- Φ is injective: $\varphi = (f_1,\dots,f_m)$, $\tilde{\varphi} = (\tilde{f}_1,\dots,\tilde{f}_m) \in Hom(V,W)$ with

$\varphi^* \equiv \tilde{\varphi}^*$ ie $g\circ\varphi = g\circ\tilde{\varphi} \in \mathbb{K}[V]$ $\forall g \in \mathbb{K}[W]$

In particular for $g = y_j$ we get $y_j\circ\varphi = f_j = y_j\circ\tilde{\varphi} = \tilde{f}_j$ on $\mathbb{K}[V]$ $\forall j$

Thus $(f_1,\dots,f_m) = (\tilde{f}_1,\dots,\tilde{f}_m) : V \longrightarrow W$ ie $\varphi = \tilde{\varphi}$.

- Φ is surjective! Fix $\psi: \mathbb{K}[W] \longrightarrow \mathbb{K}[V]$ $\mathbb{K}$-algebra homomorphism & consider

$f_j = \psi(y_j) \in \mathbb{K}[V]$ for all $j = 1,\dots,m$.

Set $\varphi = (f_1,\dots,f_m): V \to \mathbb{A}^m_{\mathbb{K}}$

Claim 1: $\varphi(\underline{p}) \in W$ $\forall \underline{p} \in V$, so $\varphi \in Hom(V,W)$

Pf/ We need to show that $\forall g \in I(W)$: $g\circ\varphi(\underline{p}) = 0$ $\forall \underline{p} \in V$

For any $g \in \mathbb{K}[W]$, we write $g = \sum_{\underline{\alpha}} a_\alpha y^{\underline{\alpha}} + I(W)$. Then:

(*) $$\psi(g) = \psi\Big(\sum_{\underline{\alpha}} a_\alpha y^{\underline{\alpha}}\Big) \underset{\substack{\downarrow \\ \psi \text{ homomorphism} \\ \text{of } \mathbb{K}\text{-algebras}}}{=} \sum_{\alpha} a_\alpha \psi(y)^{\underline{\alpha}} \underset{\substack{\downarrow \\ \psi(y_j)=f_j}}{=} \sum_{\alpha} a_\alpha \underline{f}^{\underline{\alpha}} = g(f_1,\dots,f_m) \in \mathbb{K}[V]$$
$$= g\circ\varphi$$

By construction, $\Psi(\bar{g}) = 0 \in \mathbb{K}[V]$ whenever $g \in I(W)$ because $\Psi(0) = 0$.

Thus if $\underline{p} \in V$: $\Psi(g)(p) = g(f_1(\underline{p}), \ldots, f_m(\underline{p})) = 0$

We conclude that $(f_1(\underline{p}), \ldots, f_m(\underline{p})) \in V(I(W)) = W$.

<u>Claim 2:</u> $\varphi^* = \Psi$ so $\Phi(\varphi) = \Psi$.

Pf/ By $(*)$ we get $\Psi(g) = g(f_1, \ldots, f_m) = g \circ \varphi = \varphi^*(g)$ $\forall g \in \mathbb{K}[W]$. $\square$

<u>Remark:</u> Theorem 2 is what determines morphisms between affine schemes. We define schemes by identifying the space with its coordinate ring so Φ will be defined tautologically.

⚠ Our construction is not well-adapted to local behaviors (eg holomorphic functions on connected open subsets of $\mathbb{C}$ vs. those defined on all $\mathbb{C}$). For this reason, we will like to extend the notion of a regular function to open subsets of $\mathbb{A}^n_{\mathbb{K}}$ (or subvarieties $W \subseteq \mathbb{A}^n_{\mathbb{K}}$) with respect to the Zariski topology. In doing so, we'll arrive naturally to the notion of sheaves.