Lecture XI: Rational & Regular functions

Recall: Given $V \subseteq \mathbb{A}^n_{\mathbb{K}}$, $W \subseteq \mathbb{A}^m_{\mathbb{K}}$ we define a polynomial map or a morphism $\varphi: V \longrightarrow W$ as $\varphi = (f_1, \ldots, f_m)$ for some $f_i \in \mathbb{K}[V]$ $i = 1, \ldots, m$

Theorem: $\mathrm{Hom}(V,W) = \{\varphi: V \to W \text{ polynomial map}\} \underset{\text{bij}}{\overset{\sim}{\longleftrightarrow}} \{\mathbb{K}[W] \to \mathbb{K}[V] \text{ } \mathbb{K}\text{-algebra homomorphism}\}$

$$\varphi \longmapsto \varphi^* \quad (\bar{g} \mapsto \overline{g \circ \varphi})$$

for $g \in \mathbb{K}[x_1, \ldots x_n]$

Main example: $\mathbb{K}[V] \underset{\text{bij}}{\simeq} \{\varphi: V \longrightarrow \mathbb{A}^1_{\mathbb{K}} \text{ morphism}\}$

§1. More on Morphisms:

Lemma 1: $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ & $id_V^* = id_{\mathbb{K}[V]}$.

Proof Direct from the definitions.

Definition: Two affine varieties V, W are isomorphic if $\exists$ $\varphi \in \mathrm{Hom}(V,W)$ & $\psi \in \mathrm{Hom}(W,V)$ s.t. $id_W = \varphi \circ \psi : W \longrightarrow W$ & $id_V = \psi \circ \varphi : V \longrightarrow V$.

Combining the Lemma & the Theorem we get:

Corollary 1: Two affine varieties are isomorphic if, and only if, their coordinate rings are.

Example: $\varphi: \mathbb{A}^1_{\mathbb{K}} \longrightarrow V(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{K}}$ is a homeomorphism but NOT an iso
$$t \longmapsto (t^2, t^3)$$

$\varphi^* = \mathbb{K}[y^2 - x^3] \simeq \mathbb{K}[t^2, t^3] \longrightarrow \mathbb{K}[t]$ is not an isomorphism.

Later on we'll see that these varieties have the same dimension $(=1)$.

As we discussed in Lecture X, this construction is a bit too restrictive (in analogy with holomorphic functions defined on domains of $\mathbb{C}$ rather than on all $\mathbb{C}$). For this reason, we would like to extend our notion of morphisms to a local setting, ie to open subsets U of V with respect to the Zariski Topology. In doing so, we'll arrive naturally to the notion of sheaves.

- From now on, we restrict to irreducible affine varieties, since a map $\varphi: V \longrightarrow W$ on a reducible variety $V = V_1 \cup \cdots \cup V_r$ V_i irreducible $\forall i$ correspond to r maps $\varphi_i: V_i \longrightarrow W$ that agree on the overlaps.

Definition: We say a map $\varphi: V \longrightarrow W$ is dominant if $\overline{\varphi(V)} = W$

Note: This is a better notion than surjectivity since $\varphi(V)$ need not be closed.

Lemma 2: If $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ is irreducible, then $\overline{\varphi(V)} \subseteq W \subseteq \mathbb{A}^m_{\mathbb{K}}$ is irreducible

Proof: If $\overline{\varphi(V)} = Z_1 \cup Z_2$ is a decomposition of $\overline{\varphi(V)}$ with Z_1, Z_2 closed we set $\varphi^{-1}(Z_1 \cup Z_2) = \underbrace{\varphi^{-1}(Z_1) \cup \varphi^{-1}(Z_2)}_{\text{closed}} = V$ is a decomposition of V.

Since V is irreducible, we have $V = \varphi^{-1}(Z_i)$ for some $i = 1, 2$. Thus, $\varphi(V) = \varphi(\varphi^{-1}(Z_i)) \subseteq Z_i$. Taking closure we get $\overline{\varphi(V)} \subseteq \overline{Z_i} = Z_i$.

Corollary 2: We can always reduce our study of morphisms (in particular, proofs) to the case when V & W are both irreducible & the maps $V \longrightarrow W$ are dominant.

§2. Rational functions on irreducible affine varieties

Fix $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ an irreducible affine variety, so $\mathbb{K}[W]$ is a domain.

Definition: The function field of W or the field of rational functions on W is the field $\mathbb{K}(W) := \operatorname{Quot}(\mathbb{K}[V])$

Examples: ① $\mathbb{K}(\mathbb{A}^n_{\mathbb{K}}) = \mathbb{K}(x_1, \ldots, x_n)$

② $W = \{y^2 = x^3\} \subseteq \mathbb{A}^2_{\mathbb{K}}$. then $\mathbb{K}(W) = \operatorname{Quot}(\mathbb{K}[t^2, t^3]) = \mathbb{K}(t)$

$\hookrightarrow t = t^3/t^2$

③ $W = \{y^2 = x^3 + x\} \subseteq \mathbb{A}^2_{\mathbb{K}}$ then $\mathbb{K}(W) \simeq \mathbb{K}(x)[y]/(y^2 - x^3 - x)$ is a degree 2 extension of $\mathbb{K}(x)$

(Reason : $\mathbb{K}[W] = \mathbb{K}[x][y]/(y^2-x^3-x)$] finite integral (Noether position) so $y^2 = x^3 - x \in \mathbb{K}[W]$

$\mathbb{K}[x]$ $\quad$ $\mathbb{K}(x)$ invertible

$\mathbb{K}$

$L = \mathbb{K}(x)[y]/(y^2-x^3-x)$ is a field & $\mathbb{K}[W] \subseteq L \subseteq \mathbb{K}(W)$.)

Definition: A map $\varphi: W \dashrightarrow \mathbb{A}^1_{\mathbb{K}}$ is rational if $\exists\, g, h \in \mathbb{K}[W]$ $h \neq 0$ such that $\varphi(p) = \frac{g(p)}{h(p)}$ $\forall p \in W$.

Domain of φ = $W \cap \underbrace{\{p \mid h(p) \neq 0\}}_{D(h)\text{ (basic open for the Zariski Topology on } \mathbb{A}^n_{\mathbb{K}})} = W \cap D(h)$ is dense in W

because W is irreducible & $W \cap D(h) \neq \emptyset$ is an open subset of W.

Observation: $\{\underset{\text{rational map}}{\varphi : W \dashrightarrow \mathbb{A}^1_{\mathbb{K}}}\} = \mathbb{K}(W)$.

From this we get a natural definition for rational maps between irreducible affine varieties. As in the case of morphisms, we have a wish list.

1. If V, W are irreducible varieties, then any morphism $\varphi: V \longrightarrow W$ should be a rational map.
2. Composition of rational morphisms are rational morphisms, $1_V : V \to V$ is a rational morphism
3. Compatibility with pullback maps. Rational maps must be continuous in the Zariski Topology.
4. Rational maps from W to $\mathbb{A}^1_{\mathbb{K}}$ correspond to $\mathbb{K}(W)$

There is a single possible definition with these properties.

Definition: Fix $V \subseteq \mathbb{A}^n_{\mathbb{K}}$, $W \subseteq \mathbb{A}^m_{\mathbb{K}}$ irreducible varieties. A rational function from V to W is a map $\varphi: V \dashrightarrow W$ defined on a dense open of V by a collection $g_1, \dots, g_m \in \mathbb{K}[V]$, $h_1, \dots, h_m \in \mathbb{K}[V] \setminus \{0\}$ via

$$\varphi(p) = \left(\frac{g_1(p)}{h_1(p)}, \ldots, \frac{g_m(p)}{h_m(p)} \right) \qquad \forall p \in V$$

Domain $= V \cap D(h_1) \cap \ldots \cap D(h_m) = V \cap D(h_1 \cdots h_m)$ is dense in V because $h := h_1 \cdots h_m \neq 0$ in $\mathbb{K}[V]$ (because $\mathbb{K}[V]$ is a domain) so $V \cap D(h)$ is a non-empty open in V.

- We write $\mathrm{Rat}(V,W)$ for the set of all rational functions from V to W.

Remark: From the construction we have ①, ② & ④ for free. Key part for ② is to notice that preimages of dense opens under rational maps are dense opens so $\mathrm{Dom}(f \circ g) \subseteq \mathrm{Dom}(g)$ is a dense open set of V if $g: V \dashrightarrow W$, $f: W \dashrightarrow X$ are rat'l.

Continuity is left as an exercise. Compatibility with pullbacks is a direct consequence of the next result, whose proof follows verbatim from the polynomial case:

Theorem 1: If $V \subseteq \mathbb{A}^n_{\mathbb{K}}$ & $W \subseteq \mathbb{A}^m_{\mathbb{K}}$ are irreducible affine subvarieties, then $\varphi \mapsto \varphi^*$ defines a bijection:

$$\begin{array}{ccc} \mathrm{Rat}(V,W) & \xrightarrow{\Phi} & \{ \varphi : \mathbb{K}(W) \to \mathbb{K}(V) \ \mathbb{K}\text{-algebra homomorphisms} \} \\ \varphi & \longmapsto & \varphi^* \end{array}$$

Furthermore, $(\varphi_1 \circ \varphi_2)^* = \varphi_2^* \circ \varphi_1^*$ & $(id_V)^* = id_{\mathbb{K}(X)}$.

Definition: Two irreducible affine varieties V, W are birational to each other if if $\exists\, \varphi \in \mathrm{Rat}(V,W)$ & $\psi \in \mathrm{Rat}(W,V)$ s.t. $id_W = \varphi \circ \psi : W \dashrightarrow W$ & $id_V = \psi \circ \varphi : V \dashrightarrow V$.

Corollary 3: Two irreducible affine varieties over $\mathbb{K}$ are birational if, and only if, their function fields are isomorphic as extensions of $\mathbb{K}$

⚠ Birational varieties need not be isomorphic. Our main example for this will be blow-ups.

§3. Regular functions on irreducible affine varieties

Fix $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ an irreducible affine variety, so $\mathbb{K}[W]$ is a domain.

Definition: Fix $U \subseteq W$ an open subset and let $\varphi : U \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ be a map.

(1) Given a point $p \in U$, we say that φ is regular at p if in some open neighborhood V of p with $V \subseteq U$ we can express $\varphi_{|V}$ as $\frac{g}{h}$ where $g, h \in \mathbb{K}[z_1, \ldots, z_n]$ with h is nowhere zero on V. (Note: $V \subseteq D(h) \cap U$ by construction)

(2) We say φ is regular at U if it is regular at every point of U.

- A function $\varphi : W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is regular if $\exists \{U_i\}_i \subseteq W$ open cover where $\varphi_i = \varphi_{|U_i} : U_i \to \mathbb{A}^1_{\mathbb{K}}$ is regular at U_i for all i.

- The point of the construction is to see if functions on U_i's can be "glued" to a function of f. This will only happen if they agree on the overlaps.

Lemma 3: Zero loci of regular functions to $\mathbb{A}^1_{\mathbb{K}}$ are closed.
Given $U \subseteq W$ open & $\varphi : U \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ regular, then $V(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed.

Proof At any $p \in U$, pick $p \in V_p \subseteq U$ open & $g, h \in \mathbb{K}[X]$, h nowhere zero on V_p with $\varphi_{|V} = g/h$. Then $V_p \setminus V(\varphi) = \{x \in V : g(x) \neq 0\}$ is open in V_p, hence in V. Conclude: $U \setminus V(\varphi) = \bigcup_{p \in U} (V_p \setminus V(\varphi))$ is open in U. $\square$

Corollary 4: Two regular functions $\varphi, \psi : W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ agree if and only if they agree on a dense open subset of W.

Proof $\varphi - \psi : W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ agrees with the constant zero function on a dense open set so $V(\varphi - \psi)$ is closed by Lemma 3 & contains a dense open $U \subseteq W$. Therefore

$W = \overline{U} \subseteq \overline{V(\varphi - \psi)} = V(\varphi - \psi)$, ie $\varphi = \psi$ on W.