# Lecture XII: Regular functions II

Fix $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ an irreducible affine variety, so $\mathbb{K}[W]$ is a domain.

**Recall:** A map $\varphi: W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is regular at a point $p \in W$ if $\exists\, p \in V \subseteq W$ open & $g, h \in \mathbb{K}[W]$, with $h$ nowhere zero on $V$ with $\varphi|_V = \frac{g}{h}$.

**Notation:** $\mathcal{O}_W(\underset{\text{open}}{U}) := \{\varphi : U \longrightarrow \mathbb{A}^1_{\mathbb{K}} \text{ regular at each pt of } U\}$ (sheaf inspired)

**IDEA:** Polynomial maps $\subseteq$ Rational maps $\subseteq$ Regular map = "Locally rational" provides more flexibility to study varieties locally via maps to $\mathbb{A}^1_{\mathbb{K}}$.

## §1. More on regular maps to $\mathbb{A}^1_{\mathbb{K}}$:

**Proposition 1:** A regular function $\varphi: W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is continuous in the Zariski Topology

**Proof:** We show that $\varphi^{-1}(V)$ is closed in $W$ for each $V \subseteq \mathbb{A}^1_{\mathbb{K}}$ closed. By construction, $V$ is a finite so it is enough to show $\varphi^{-1}(a) \subseteq W$ is closed for any $a \in \mathbb{A}^1_{\mathbb{K}}$. This property can be checked locally! Indeed:

**Fact:** A subset $Z$ of a topological space $Y$ is closed if, and only if, for any open cover $\{U_i\}_i$ of $Y$ we have that $Z \cap U_i$ is closed in $U_i$ for each $i$.

- Pick a cover $\{U_i\}$ of $W$ so that $\varphi|_{U_i}$ is regular & we can write $\varphi|_{U_i} = \frac{g_i}{h_i}$ where $g_i, h_i \in \mathbb{K}[W]$ & $h_i$ is nowhere $0$ on $U_i$. Then:

$$\varphi^{-1}(a) \cap U_i = \left\{ p \in U_i \;\middle|\; \frac{g_i(p)}{h_i(p)} = a \right\} = \{ p \in U_i \mid g_i(p) - a h_i(p) = 0\}$$

$$= U_i \cap \underbrace{V(g_i(\underline{x}) - a\, h_i(\underline{x}))}_{\text{closed in } \mathbb{A}^n_{\mathbb{K}}} \quad \text{is closed in } U_i.$$

(it's complement in $U_i$ is open)

**Conclude:** $\varphi^{-1}(a)$ is closed in $W$. □

This statement is consistent with the following fact proved in Lecture XI.

**Lemma 1:** Zero loci of regular functions to $\mathbb{A}^1_{\mathbb{K}}$ are closed.

**Examples:** ① Polynomial functions are regular

② $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto \frac{1}{1+x^2}$ is regular, but not polynomial!

③ $\varphi: \mathbb{C} \to \mathbb{C}$ regular must be a polynomial

Why? Since $\mathbb{C} = \mathbb{A}^1_{\mathbb{C}}$ is Noetherian (HW1), any open cover of $\mathbb{C}$ admits a finite subcover.

In particular, we can find $U_1, \ldots, U_s$ & polynomials $g_1, \ldots, g_s$ & $h_1, \ldots, h_s$ such that $\varphi|_{U_i} = \dfrac{g_i}{h_i}$ for $i = 1, \ldots, s$. & $h_i$ is nowhere vanishing on $U_i$

By construction $U_i = \mathbb{C} \smallsetminus W_i$ $W_i \subseteq \mathbb{C}$ closed, so $W_i$ = finite collection of points

Since $\mathbb{C}$ is algebraically closed, $h_i = \prod_{j=1}^{m_i} (x - p_j^{(i)})$ with $W_i = \{p_1^{(i)}, \ldots, p_{m_i}^{(i)}\}$

On the overlaps, we have $\varphi|_{U_i \cap U_\ell} = \dfrac{g_i}{h_i} = \dfrac{g_\ell}{h_\ell}$ on a dense open of $\mathbb{C}$.

Equivalently, $g_i h_\ell = h_i g_\ell$ on a dense open of $\mathbb{C}$, so $g_i h_\ell = h_i g_\ell$ on $\mathbb{C}[x]$

We can assume $\gcd(g_i, h_i) = 1 = \gcd(g_\ell, h_\ell)$ so this means

$h_i | h_\ell$ & $g_\ell | g_i$ on $\mathbb{C}[x]$.

In other words $\dfrac{g_i}{h_i} = \dfrac{g_\ell}{h_\ell}$ on $\mathbb{C}(x)$ for all $i, \ell$.

Conclusion: $\exists\, g, h \in \mathbb{C}[x]$ with $\gcd(g, h) = 1$ such that $f = \dfrac{g}{h}$.
& $h$ is nowhere vanishing on $\mathbb{C}$.

By the Nullstellensatz $V(\langle h \rangle) = \emptyset$ on $\mathbb{C}$ forces $\langle h \rangle = 1$. Thus, $f$ is a polynomial.

Remark: Example 3 illustrates another advantage of working with algebraically closed fields!

## §2. Regular maps between irreducible varieties:

The definition of regular maps extends naturally to maps between irreducible varieties
Regular functions are "locally rational functions". More precisely,

Definition: Given two irreducible varieties $V, W$ a map $\varphi: V \to W$ is regular if (1) $\varphi$ is continuous;

(2) for every open $U \subseteq W$ & every regular function $f: U \to \mathbb{A}^1_{\mathbb{K}}$, the function

$f \circ \varphi : \varphi^{-1}(U) \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is regular.

Note: This type of definition is typically of AG (a map has a given property (P) if when composing with all maps to $\mathbb{A}^1_{\mathbb{K}}$ (or $\mathbb{P}^1_{\mathbb{K}}$) with property (P), the resulting map to $\mathbb{A}^1_{\mathbb{K}}$ has property (P)).

Proposition 2: Composition of regular maps is regular ; $1_V : V \longrightarrow V$ is regular.

- Regularity gives us a better definition of rational maps. (see HW 4)

Definition: A rational map $\varphi : V \dashrightarrow W$ between two irreducible varieties is defined to be an equivalence class of pairs $(U, \gamma)$ where

(1) $U \subseteq V$ is a Zariski dense open &

(2) $\gamma : U \longrightarrow W$ is a regular map

Here, $(U, \gamma) \sim (U', \gamma')$ if, and only if, $\gamma|_{U \cap U'} = \gamma'|_{U \cap U'}$.

⚠ Not all regular functions to $\mathbb{A}^1_{\mathbb{K}}$ are global quotients of 2 polynomials

Example: $W = V(xw - yz) \subseteq \mathbb{A}^4_{\mathbb{K}}$ & $U = W \smallsetminus V(y,w) \subseteq W$

Then $\varphi : U \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ $\quad (x,y,z,w) \longmapsto \begin{cases} x/y & \text{if } y \neq 0 \\ z/w & \text{if } w \neq 0 \end{cases}$ is regular on $U$.

- Well-defined: if $yw \neq 0$ then $\frac{x}{y} = \frac{z}{w}$ (because $xw - yz = 0$)

- If $\varphi = \frac{g}{h}$ on $U$ with $g, h \in \mathbb{K}[W]$ & $h$ nowhere vanishing on $U$

  - $\frac{x}{y} = \frac{g}{h}$ on $y \neq 0$ $\Rightarrow$ $xh = gy$ means $y | h$
  - $\frac{z}{w} = \frac{g}{h}$ on $w \neq 0$ $\Rightarrow$ $zh = gw$ —— $w | h$

Conclusion: $yw | h$ & $h$ will vanish on $(\underset{x}{*}, \underset{y}{0}, \underset{z}{0}, \underset{w}{*}) \in W$ & $(\underset{x}{*}, \underset{y}{*}, \underset{z}{0}, \underset{w}{0}) \in W$.

so the expression will fail to be regular on $(D_W(z) \cup D_W(w)) \cap U$.

Key $\mathbb{K}[W]$ is not a UFD.

## §3. Algebraic structure of regular functions:

Next, we focus on the algebraic properties of regular functions:

Lemma 2: Regular functions $\varphi : W \to \mathbb{A}^1_{\mathbb{K}}$ at a fixed point $p$ form a ring.
We denote this ring by $\mathcal{O}_p$. Furthermore, $\mathcal{O}_p$ is a local ring, with maximal ideal $m_p = \{\varphi \in \mathcal{O}_p \quad \varphi(p) = 0\}$

Proof: It is easy to see that $\varphi$ is a ring with operations pointwise addition & multiplication. Constant maps $0$ & $1$ are the neutral elements in $\mathcal{O}_p$.

- Now, $m_p$ is a proper ideal of $\mathcal{O}_p$. To show it's maximal we need to check any $\varphi \in \mathcal{O}_p \setminus m_p$. is invertible in $\mathcal{O}_p$.

- We write $\varphi$ as $\varphi : U \to \mathbb{A}^1_{\mathbb{K}}$ for $p \in U$ open with $\varphi = \frac{g}{h}$ for $g, h \in \mathbb{K}[W]$ & $h$ nowhere vanishing in $U$. In particular $g(p) \neq 0$ so we have $g$ is nowhere vanishing on $U \cap D(g) = V$ & $p \in V$.

Conclusion: $\frac{1}{\varphi} = \frac{h}{g} : V \to \mathbb{A}^1_{\mathbb{K}}$ is regular at $V$ so $\frac{1}{\varphi} \in \mathcal{O}_p$. $\square$

Lemma 3: Regular functions $\varphi : W \to \mathbb{A}^1_{\mathbb{K}}$ form a ring (denote it by $\mathcal{O}(W)$ or $\mathcal{O}_W(W)$)

Corollary 2: Restrictions induce a natural map of rings for each $p \in W$

$$\mathcal{O}(W) \longrightarrow \mathcal{O}_p \longrightarrow \mathbb{K}(W)$$

These maps are injective because $W$ is irreducible.

- As we saw earlier, regular functions are more restricted when $\overline{\mathbb{K}} = \mathbb{K}$ because the nowhere vanishing condition can be controlled via a (relative) Nullstellensatz

**Theorem**: Fix $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ irreducible & assume $\overline{\mathbb{K}} = \mathbb{K}$. Then:

(1) $\mathbb{K}[W] \cong \mathcal{O}(W) \subseteq \mathbb{K}(W)$ via $f \longmapsto \frac{f}{1}$.

(2) The map $W \longrightarrow \{\text{Max ideals of } \mathbb{K}[W]\}$, $p \longmapsto m_p$ is a bijection.

(3) [Regular functions at $p$ correspond to localizations at $m_p$]
For each $p$, $\mathcal{O}_p \simeq \mathbb{K}[W]_{m_p}$

(We'll see a proof next time)

**Key**: Characterize regular functions on basic affine opens

**Proposition 3**: Fix $W$ irreducible & $f \in \mathbb{K}[W]$, then:

$$\mathcal{O}_W(D(f)) = \left\{ \frac{g}{f^n} \quad g \in \mathbb{K}[W],\ n \in \mathbb{Z}_{\geq 0} \right\}$$

In particular, for $f = 1$ we get $\mathcal{O}_W(W) = \mathcal{O}(W) = \mathbb{K}[W] \subseteq \mathbb{K}(W)$.
$g \longmapsto \frac{g}{1}$

⚠ This statement fails for arbitrary opens ( **Example** (HW 4): $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \smallsetminus \{(0,0)\}) = \mathbb{K}[\mathbb{A}^2]$)

**Corollary 3**: On basic opens, regular functions are global rational functions