# Lecture XIII: Regular functions & sheaves

## §1. Regular functions when $\overline{\mathbb{K}}=\mathbb{K}$:

Our next goal is to prove the following theorem characterizing regular functions when $\overline{\mathbb{K}}=\mathbb{K}$

**Theorem:** Fix $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ irreducible & assume $\overline{\mathbb{K}}=\mathbb{K}$. Then:

(1) $\mathbb{K}[W] \simeq \mathcal{O}(W) \subseteq \mathbb{K}(W)$ via $f \longmapsto \frac{f}{1}$.

(2) The map $\begin{array}{ccc} W & \longrightarrow & \{\text{Max ideals of } \mathbb{K}[W]\} \\ p & \longmapsto & m_p \end{array}$ is a bijection.

(3) [Regular functions at $p$ correspond to localizations at $m_p$]
For each $p$, $\mathcal{O}_p \simeq \mathbb{K}[W]_{m_p}$

**Remark:** The statement fails when $\overline{\mathbb{K}}\neq\mathbb{K}$. For example $\varphi(x)=\frac{1}{x^2+1}: \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ lies in $\mathcal{O}(\mathbb{A}^1_{\mathbb{R}}) \smallsetminus \mathbb{R}[\mathbb{A}^1_{\mathbb{R}}]$.

**Proof:** (2) follows from (1) and the characterization of maximal ideals of $\mathbb{K}[W]$ used to prove the weak Nullstellensatz.

For (1) and (3) we need the following result:

**Proposition 1:** Fix $W$ irreducible & $f \in \mathbb{K}[W]$, then:

$$\mathcal{O}_W(D(f)) = \left\{ \frac{g}{f^n} \quad g\in \mathbb{K}[W],\ n\in \mathbb{Z}_{\geq 0} \right\}$$

In particular, for $f=1$ we get $\mathcal{O}_W(W) = \mathcal{O}(W) = \mathbb{K}[W] \subseteq \mathbb{K}(W)$.
$g \longmapsto \frac{g}{1}$

**Proof:** We prove the double inclusion

($\supseteq$) is clear because $f^n$ is nowhere vanishing on $D(f)$.

($\subseteq$) Fix $\varphi \in \mathcal{O}_W(D(f))$. We need to show that any local representation of $\varphi$ can be made to have the form $g/f^n$ for some $g\in \mathbb{K}[V]$ & $n\in\mathbb{Z}_{\geq 0}$. For this we need to adjust our given local presentations.

Fix $p\in D(f)$. Since $\varphi$ is regular at $p$, $\exists$ open $V$ with $p\in V\subseteq D(f)$ &

$g_p, h_p \in \mathbb{K}[W]$ with $h_p$ nowhere vanishing on $V$ s.t $\varphi|_V = \frac{g_p}{h_p}$.

We can shrink $V$ to a basic open in $D(f)$ containing $p$. Call it $V = D(f_p)$


We have (1) $\frac{g_p}{h_p} = \frac{g_p f_p}{h_p f_p}$ on $D(f_p)$

(2) $h_p f_p$ is nowhere vanishing on $D(f_p)$.

(3) $p \in D(h_p f_p) = D(f_p) \subseteq D(f)$.

Conclusion: we can replace $g_p$ by $g_p f_p$ & $h_p$ by $h_p f_p$ & assume $\varphi = \frac{g_p}{h_p}$ on $D(h_p)$.

(*) Advantage: The new $g_p$ & $h_p$ both vanish on $V(h_p) \cap D(f) = D(f) \setminus D(h_p)$.

· To finish, we need to find $g \in \mathbb{K}[W]$ & $n \in \mathbb{Z}_{\geq 0}$ with $\frac{g_p}{h_p} = \frac{g}{f^n}$ $\forall p$.

Claim 1: $f \in \sqrt{\langle h_p : p \in D_W(f) \rangle}$

Pf/ The collection $\mathcal{D} := \{D(h_p)\}_{p \in D(f)}$ is an open cover of $D(f)$ so

$$D(f) = \bigcup_{p \in D(f)} D(h_p).$$

Taking the complement in $W$, we get $V_W(f) = \bigcap_{p \in D(f)} V_W(h_p) = V_W(\langle h_p : p \in D(f) \rangle)$

Since $\overline{\mathbb{K}} = \mathbb{K}$, the relative Hilbert's Nullstellensatz says:

$$f \in I_W(V_W(f)) = I_W(V_W(\langle h_p : p \in D(f) \rangle)) = \sqrt{\langle h_p : p \in D(f) \rangle}$$

So $\exists\, n \in \mathbb{Z}_{\geq 0}$ & finitely many $p_1, \dots, p_s$ & $k_i \in \mathbb{K}[W]$ for $i = 1, \dots, s$ with

$\boxed{f^n = \sum_{i=1}^{s} k_i h_{p_i}}$ . Set $\boxed{g := \sum_{i=1}^{s} k_i g_{p_i}}$

Claim 2: $\varphi = \frac{g}{f^n}$. (Equivalently, $\frac{g}{f^n} = \frac{g_p}{h_p}$ on $D(h_p)$ $\Leftrightarrow$ $g h_p = g_p f^n$ on $D(h_p)$) for all $p \in D(f)$

Pf/ We first relate the presentations of $\varphi$ on the overlaps of the open cover $\mathcal{D}$ of $D(f)$

Pick $p, p' \in D(f)$ & assume $\mathcal{U} := D(h_p) \cap D(h_{p'}) \neq \emptyset$. By construction:

$$\frac{g_p}{h_p} = \frac{g_{p'}}{h_{p'}} \quad \text{on } \mathcal{U}.$$

**Note:**
- $g_p h_{p'} = g_{p'} h_p$ on $\mathcal{U}$
- $g_p h_{p'}$ vanishes on $Y := V_{D(f)}(h_p) \cup V_{D(f)}(h_{p'})$ by (*)
- $g_{p'} h_p$

**Consequence:** $g_p h_{p'} = g_{p'} h_p$ on $D(f) = \mathcal{U} \cup Y$

To finish: $g h_p = \sum_{i=1}^{s} k_i \underbrace{g_{p_i} h_p} = \sum_{i=1}^{s} k_i \underbrace{g_p h_{p_i}} = g_p \sum_{i=1}^{s} k_i h_{p_i} = g_p f^n$

on $D(f)$, hence this is true on $D(h_p)$ for all $p \in D(f)$. $\square$

**Corollary 1:** Regular functions on irreducible affine varieties over $\overline{\mathbb{K}} = \mathbb{K}$ are localizations

Given $W \subseteq \mathbb{A}^n_{\mathbb{K}}$ irreducible with $\overline{\mathbb{K}} = \mathbb{K}$ & any $f \in \mathbb{K}[W]$, then $\mathcal{O}_W(D(f))$ is isomorphic (as a $\mathbb{K}$-algebra) to the localization of $\mathbb{K}[W]$ at the multiplicatively closed set $\{f^n : n \in \mathbb{Z}_{\geq 0}\}$.

**Proof:**
$$\begin{array}{ccc} \mathbb{K}[W]_f & \xrightarrow{\alpha} & \mathcal{O}_W(D(f)) \\ \frac{g}{f^n} & \longmapsto & \frac{g}{f^n} \end{array}$$
is well-defined because $\mathbb{K}[W]$ is a domain.

- $\alpha$ is a $\mathbb{K}$-algebra homomorphism by construction.
- $\alpha$ is injective by construction.
- $\alpha$ is surjective by Proposition 1. $\square$

- Finally, we prove part (3) of the Theorem.

Recall we have natural maps
$$\begin{array}{ccccc} \mathbb{K}[W] = \mathcal{O}(W) & \longrightarrow & \mathcal{O}_p & \longrightarrow & \mathbb{K}(W) \\ f & \longmapsto & \frac{f}{1} & & \\ & & \frac{g}{h} & \longmapsto & \frac{g}{h} \end{array}$$
where $h$ nowhere $0$ on a nbhd of $p$.

**Note:** $m_p \subseteq \mathcal{O}(W) = \mathbb{K}[W]$ is a maximal ideal & any $\varphi \in \mathcal{O}(W) \setminus m_p$ is a unit in $\mathcal{O}_p$.

Thus, we get a map $\mathbb{K}[W]_{m_p} = \mathcal{O}(W)_{m_p} \xrightarrow{\Phi} \mathcal{O}_p$

$\frac{g}{h} \longmapsto \frac{g}{h}$ for any $g, h \in \mathbb{K}[W]$ & $h \notin m_p$.

- $\Phi$ is surjective: $\frac{g}{h} \in \mathcal{O}_p \iff \exists\, f \in \mathbb{K}[W]$ with $f(p) \neq 0$ & $g', h' \in \mathbb{K}[W]$

$\hookrightarrow$ so $p \in D(f) \subseteq W$

such that $\frac{g}{h}\Big|_{D(f)} = \frac{g'}{h'}\Big|_{D(f)}$ as regular functions with $h'$ nowhere zero on $D(f)$.

Since $h(p), h'(p) \neq 0$ we conclude $h, h' \notin m_p$ so $\frac{g}{h} \in \mathbb{K}[W]_{m_p}$.

- $\Phi$ is injective: $\Phi(\frac{g}{h}) = 0 \in \mathcal{O}_p$ with $h \notin m_p$ $\iff \exists\, f \in \mathbb{K}[W]$ with $f(p) \neq 0$ such that

$h$ is nowhere $0$ on $D(f)$ & $\frac{g}{h}\Big|_{D(f)} = 0\Big|_{D(f)}$

Since $h$ is nowhere $0$, this last condition is equivalent to $g|_{D(f)} = (g - 0 \cdot h)|_{D(f)} = 0$ as function on $D(f)$. Since $g, 0 \in \mathbb{K}[W]$ are continuous functions on $W$ and agree on a dense open subset of $W$, we conclude that $g = 0$ on $W$, ie $g = 0 \in \mathbb{K}[W]$, so $\frac{g}{h} = 0 \in \mathbb{K}[W]_{m_p}$.

## §2. Sheaf Theory:

Fix $X$ topological space & $\mathcal{C}$ a category (Sets, Ab = abelian gps, Rings, $\mathrm{Mod}_R$ = modules over a commutative ring $R$, $\mathrm{Vect}_{\mathbb{K}}$ = $\mathbb{K}$-vector spaces, etc)

**Definition:** A *presheaf* $\mathcal{F}$ of objects in $\mathcal{C}$ on the space $X$ is a functor $(\mathrm{Op}(X))^{op} \longrightarrow \mathcal{C}$ where $\mathrm{Op}(X)$ = category of opens of $X$

Objects = $\{U \subseteq X \text{ open}\}$ & $\mathrm{Hom}(U,V) = \begin{cases} * & \text{if } U \subseteq V \text{ (corresponding to the inclusion)} \\ \emptyset & \text{otherwise} \end{cases}$

More concretely, a *presheaf* on $X$ with values in $\mathcal{C}$ is a pair $(\mathcal{F}, \rho)$ where:

(1) $\mathcal{F}$ is an assignment: $U \subseteq X$ open $\longmapsto \underbrace{\mathcal{F}(U)}_{=\Gamma(U,\mathcal{F})} \in \mathrm{Obj}(\mathcal{C})$. (sections on $U$)

(2) For each pair $V \subseteq U \subseteq X$ of opens, we have $\mathcal{F}(U) \xrightarrow[\text{(restriction map)}]{\rho_{U,V}} \mathcal{F}(V) \in \mathrm{Homs}(\mathcal{C})$

satisfying the following properties:

(i) $\rho_{U,U} = id_{\mathcal{F}(U)} \quad \forall\, U \subseteq X$ open

(ii) For each triple $W \subseteq V \subseteq U$ of opens in $X$ we have

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{U,V}"] \arrow[rr, bend right, "\rho_{U,W}"'] & \mathcal{F}(V) \arrow[r, "\rho_{V,W}"] & \mathcal{F}(W) \end{tikzcd} \qquad \rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$$

We omit $\rho$ when understood from context and write $\boxed{f|_V := \rho_{U,V}(f)}$ for $f \in \mathcal{F}(U)$.

**Example** For $U \subseteq \mathbb{C}$ open $\mapsto \mathcal{F}(U) = \{f : U \to \mathbb{C},\ f \text{ is holomorphic}\}$

For $V \subseteq U \subseteq \mathbb{C}$ opens: $\rho_{U,V}$ = usual restrictions.

$(\mathcal{F}, \rho)$ is a presheaf of vector spaces over $\mathbb{C}$ / or of sets.

**Definition**: A presheaf $(\mathcal{F}, \rho)$ is a sheaf if the following local gluing axiom holds (cocycle condition)

" For every open $U$ of $X$ & every open cover $\{U_i\}_{i\in I}$ of $U$ $\left(U = \bigcup_{i\in I} U_i\right)$ with $f_i \in \mathcal{F}(U_i)\ \forall i$ satisfying $\rho_{U_i, U_i\cap U_j}(f_i) = \rho_{U_j, U_i \cap U_j}(f_j)\ \forall i,j$ ($f_i|_{U_i\cap U_j} = f_j|_{U_i \cap U_j}$, ie agreement on the overlaps) we have a unique $f \in \mathcal{F}(U)$ with $\rho_{U,U_i}(f) = f_i \quad \forall i \in I$ "

Equivalently, if $\forall\, U \in \mathcal{O}p(X)$ & any covering $\{U_i\}_{i\in I}$ of $U$, the sequence:

$$\mathcal{F}(U) \xrightarrow{a = \prod_i \rho_{UU_i}} \prod_{i\in I} \mathcal{F}(U_i) \overset{\rho_1^*}{\underset{\rho_2^*}{\rightrightarrows}} \prod_{(i,j)} \mathcal{F}(U_i \cap U_j)$$

is exact. That is, $a$ is injective & if $\eta \in \prod_i \mathcal{F}(U_i)$ st. $\rho_1^*(\eta) = \rho_2^*(\eta)$, then $\eta \in Im(a)$

**Note**: If $\mathcal{C}$ = Ab group we can equivalently require the exactness of

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \xrightarrow{\rho_1^* - \rho_2^*} \prod_{(i,j)} \mathcal{F}(U_i \cap U_j)$$

**Remark**: Working with $U = \emptyset$ & the empty covering we see that $\mathcal{F}(\emptyset)$ is the terminal object in the category $\mathcal{C}$ (an object in $\mathcal{C}$ has a unique map to it: $\rho_{U\emptyset} : \mathcal{F}(U) \to \mathcal{F}(\emptyset)$)

<u>Examples</u>: ① $X, Y$ Topological spaces

$\mathcal{F}$ = presheaf on $X$ given by $\mathcal{F}(U) = \{ f: U \to Y \mid f \text{ is continuous} \} \in \mathrm{Set}$

$\forall\, V \subseteq U$ opens $\quad \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is the usual restriction map
$$g \longmapsto g|_V$$

<u>Claim</u>: $\mathcal{F}$ is a sheaf

Pf/ Fix $U \subseteq X$ open & $\{U_i\}_{i\in I}$ an open cover of $U$. Consider $g_i \in \mathcal{F}(U_i)$ satisfying the cocycle condition:

(1) Lifting property: For each $p \in U$ define $g$ via $g_i(p)$ when $p \in U_i$

- <u>Well-defined</u>: if $p \in U_i \cap U_j$ then $g_i|_{U_i\cap U_j} = g_j|_{U_i \cap U_j}$ so $g_i(p) = g_j(p)$.
- <u>Continuity</u> $g|_{U_i} = g_i$ & $g_i$ is continuous, so $g$ is continuous as well.

(2) Uniqueness: if $g_1, g_2 : U \longrightarrow Y$ are such that $g_1|_{U_i} = g_2|_{U_i}$ for all $i \in I$, then by definition of a function we have that $g_1 = g_2$ (each $p \in U$ lies in some $U_i$)

② Same construction works for $\mathcal{C}^\infty$-manifolds, complex manifolds, etc.

③ $A$ abelian gp $\neq \{0\}$, $X$ locally connected topological space

- $\mathcal{F}$ = presheaf of constant functions from $X$ to $A$.

  $\mathcal{F}(U) = A \quad \forall\, U \subseteq X$ open, $U \neq \emptyset$. & $\mathcal{F}(\emptyset) = \{0\}$

- $\mathcal{F}$ is <u>not</u> a sheaf. <u>Reason</u>: Pick $U = U_1 \sqcup U_2$ & $a_1, a_2 \in A$ distinct.

Then $a_i : U_i \to A \in \mathcal{F}(U_i)$, $a_1|_{U_1\cap U_2} = a_2|_{U_1 \cap U_2}$ but $\nexists\, f: U \to A$ constant with $f|_{U_i} = a_i \ \forall i$

- We can fix this by considering the presheaf $\mathcal{G}$ of locally constant functions to $A$.

Equivalently, endowing $A$ with the discrete topology, we have $\mathcal{G}(U) = \{ f : U \to A \text{ cont.} \}$ $\forall\, U \subseteq X$ open.

We saw in ① that this was a sheaf.

<u>Remark</u>: Next time will see how to generalize this construction to obtain a sheaf from a presheaf. The process will be called <u>sheafification</u>.

MAIN examples for us: $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ irreducible affine variety

1. $\mathcal{P}$ = presheaf of morphisms to $\mathbb{A}^1_{\mathbb{K}}$

$\mathcal{P}(U) = \{ \varphi : U \longrightarrow \mathbb{A}^1_{\mathbb{K}} \text{ polynomial map} \}$ for $U \subseteq X$ Zariski open with usual restriction maps.

Sheaf axiom is valid: 2 polynomials that agree on a Zariski dense open of $X$, must agree on $X$ by continuity so they give the same element of $\mathbb{K}[X]$

2. $\mathcal{R}at$ = presheaf of rational functions to $\mathbb{A}^1_{\mathbb{K}}$

$\mathcal{R}at(U) = \{ \varphi : U \dashrightarrow \mathbb{A}^1_{\mathbb{K}} \text{ rational function} \}$ is a presheaf but <u>not</u> a sheaf (locally rational $\not\Rightarrow$ globally rational, as we saw in Example §12.2)

3. $\mathcal{O}_X$ = sheaf of regular functions on $X$.

4. $\mathcal{O}_p$ = sheaf of regular functions at a pt $p$ of $X$ $\left( \mathcal{O}_p(U) = \emptyset \text{ if } p \notin U \right)$